Turnpike analysis via dynamical system approach

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Key words of the talk: Invariant manifold

- Optimal control & applications for mechatronics control
  Self-introduction, stable manifold method

- What is turnpike?
  History, econometrics, & definition

- Invariant manifolds (stable & unstable manifolds)

- Turnpike analysis via invariant manifold theory
  Lambda lemma, hyperbolicity
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Optimal control and HJE

\[
\begin{align*}
    \dot{x} &= f(x) + g(x)u, \quad x(0) = x_0 \\
    J &= \frac{1}{2} \int_0^\infty \left( x^T Q x + u^T R u \right) dt,
\end{align*}
\]

Stable regulator problem

Find a control strategy $u$ such that the system is stabilized and the cost $J$ is minimized.
HJ and optimal control

\[ \begin{align*}
\dot{x} &= f(x) + g(x)u, \quad x(0) = x_0 \\
J &= \frac{1}{2} \int_0^\infty \left( x^T Q x + u^T R u \right) \, dt,
\end{align*} \]

Solution of the stable regulator problem is given by

\[ u^* = -\frac{1}{2} R^{-1} g(x)^T \frac{\partial V}{\partial x} \]

using a solution of a pde

\[ \text{(HJ)} \quad \frac{\partial V}{\partial x} f(x) - \frac{1}{4} \frac{\partial V}{\partial x} g(x) R^{-1} g(x)^T \frac{\partial V}{\partial x} + h(x)^T h(x) = 0 \]

(Bellman’s Dynamic Programming)
(HJ) \[ H(x, p) = p^T f(x) - \frac{1}{2} p^T \tilde{R}(x) p + \frac{1}{2} x^T Q x = 0 \]

where \( \tilde{R}(x) = g(x) R^{-1} g(x)^T \)

In the linear case, (HJ) reduces to the algebraic Riccati equation

\[ PA + A^T P - PRP + Q = 0 \]
HJ and optimal control

(HJ) \[ H(x, p) = p^T f(x) - \frac{1}{2} p^T \bar{R}(x)p + \frac{1}{2} x^T Qx = 0 \]
\[ p_1 = \frac{\partial V}{\partial x_1}, \ldots, p_n = \frac{\partial V}{\partial x_n} \]

By the theory of 1st order pdes, it is equivalent to consider a Hamiltonian system

(Ham) \[ \begin{cases} 
\dot{x} = \frac{\partial H}{\partial p} \\
\dot{p} = -\frac{\partial H}{\partial x}
\end{cases} \]
✓ This result is known since ‘60.

✓ The solution methods for HJ were until recently limited
  • Taylor expansion (Al’brekht ’61, Lukes ’69)
  • Galerkin approximation (Beard ‘97)
  • …

✓ No applications had been reported except for simple numerical examples.

✓ In 2008, A. J. van der Schaft and I proposed the stable manifold method for HJ.
Stable manifold method

\[(\text{HJ}) \quad H(x, p) = p^T f(x) - \frac{1}{2} p^T \bar{R}(x) p + \frac{1}{2} x^T Q x = 0\]

\[p_1 = \frac{\partial V}{\partial x_1}, \ldots, p_n = \frac{\partial V}{\partial x_n}\]

\[(\text{Ham}) : \begin{cases}
\dot{x} = \frac{\partial H}{\partial p} \\
\dot{p} = -\frac{\partial H}{\partial x}
\end{cases}\]

The stable manifold method for (HJ)

✓ Hamiltonian ode instead of pde
✓ Invariant manifold of the Hamiltonian system instead of obtaining V
✓ Iterative computational algorithm
✓ Easier to implement
Mechatronics control via optimal control
Swing up control of flexible inverted pendulum
Swing up & stabilization of flexible IP

Light weight
Agility
Nonlinear control of flexible structure

Spring steel flexible beam
θ encoder
ϕ encoder
DC motor
Swing up & stabilization of flexible IP

Controller structure

\[ \dot{x} = f(x) + g(x)u, \ x(0) = x_0 \]

\[ \dot{x}_F = \frac{d}{dt} \begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} f(x) + g(x)u \\ -\omega u \end{bmatrix} + \begin{bmatrix} 0 \\ \omega \end{bmatrix} \nu \]
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Turnpike theory in econometrics

Dorfman, Samuelson, Solow 1958
McKenzie 1963
"if origin and destination are far enough apart, it will always pay to get on the turnpike and cover distance at the best rate of travel ...“ (Dorfman, Samuelson, Solow 1958)
Turnpike theory in control theory

- Wilde & Kokotovic 1972: Dichotomy
- Rockafellar 1973: Saddle point
- Anderson & Kokotovic 1987: nonlinear systems

- Porretta & Zuazua 2013: Infinite dim & Turnpike inequality
- Grune et.al. 2013: Dissipative system theory, MPC
- Trélat & Zuazua 2015: nonlinear systems
- Zuazua 2017: Wave equation
- Trélat, Zhang & Zuazua 2018: Periodic turnpike

$$|u^*(t) - \bar{u}| + |x^*(t, x_0) - \bar{x}| \leq K \left[ e^{-\mu t} + e^{-\mu (T-t)} \right]$$
Nonlinear optimal control problems

\[(\Sigma) \quad \dot{x} = f(x) + g(x)u, \quad x(t_0) = x_0\]

\[(OCP_1)_T : \quad \text{Find a control } u \in L^\infty(0, T; \mathbb{R}^m) \text{ such that}
\]

\[J(u) = \frac{1}{2} \int_0^T |Cx(t) - z|^2 + |u(t)|^2 \, dt\]

along \((\Sigma)\) is minimized over all \(u \in L^\infty(0, T, \mathbb{R}^m)\).

\[(OCP_2)_T : \quad \text{Find a control } u \in L^\infty(0, T; \mathbb{R}^m) \text{ such that}
\]

\[J(u) = \frac{1}{2} \int_0^T x(t)^\top C^\top Cx(t) + |u|^2 \, dt\]

along \((\Sigma)\) is minimized over all \(u \in L^\infty(0, T; \mathbb{R}^m)\) such that \(x(T) = x_1\).
Definition: turnpike

An OPC problem has the turnpike property if for any $\varepsilon > 0$, there exists an $\eta_\varepsilon > 0$ such that

$$|\{t \geq 0 \mid |u^*(t) - \bar{u}| + |x^*(t, x_0) - \bar{x}| > \varepsilon\}| < \eta_\varepsilon$$

for all $T > 0$, where $\eta_\varepsilon$ depends only on $\varepsilon$, $f$, $g$, $x_0$, and $L$ and $| \cdot |$ denotes length (Lebesgue measure) of interval.

$(\bar{u}, \bar{x})$ : steady optimal sol.

$(\bar{u}, \bar{x}) := \text{argmin} \ L(x, u) \text{ s.t. } 0 = f(x) + g(x)u$
Definition: turnpike

An OPC problem has the turnpike property if for any $\varepsilon > 0$, there exists an $\eta_\varepsilon > 0$ such that

$$\left| \left\{ t \geq 0 \mid |u^*(t) - \bar{u}| + |x^*(t, x_0) - \bar{x}| > \varepsilon \right\} \right| < \eta_\varepsilon$$

for all $T > 0$, where $\eta_\varepsilon$ depends only on $\varepsilon$, $f$, $g$, $x_0$, and $L$ and $| \cdot |$ denotes length (Lebesgue measure) of interval.

A sufficient condition for turnpike:

$$|u^*(t) - \bar{u}| + |x^*(t, x_0) - \bar{x}| \leq K \left[ e^{-\mu t} + e^{-\mu(T-t)} \right]$$

Turnpike inequality (Porretta & Zuazua 2013)
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· What is turnpike?
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Invariant manifold theory
Invariant manifold theory

Finite dimensional ode:

\[ \dot{x} = Ax + X(x, y, z) \]
\[ \dot{y} = By + Y(x, y, z) \]
\[ \dot{z} = Cz + Z(x, y, z) \]

- \( A \in \mathbb{R}^{n_x \times n_x}, \ Re\lambda(A) < 0 \) (\( x \): stable part)
- \( B \in \mathbb{R}^{n_y \times n_y}, \ Re\lambda(B) = 0 \) (\( y \): center part)
- \( C \in \mathbb{R}^{n_z \times n_z}, \ Re\lambda(C) > 0 \) (\( z \): unstable part)
- The functions \( X, Y, Z \) are continuously differentiable
- \( X, Y, Z \) together with all of their first derivatives vanish at the origin.
Invariant manifold theory

\[ \dot{x} = Ax + X(x, y, z) \]
\[ \dot{y} = By + Y(x, y, z) \]
\[ \dot{z} = Cz + Z(x, y, z) \]

Kelly (1967) shows that there exist invariant manifolds

Stable manifold \( y = v^+(x), \ z = w^+(x) \)
Center manifold \( x = u^*(y), \ z = w^*(y) \)
Unstable manifold \( x = u^-(z), \ y = v^-(z) \)
Center stable manifold \( z = w^{*+}(x, y) \)
Center unstable manifold \( x = u^{*-}(y, z) \)
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Iterative algorithms for computation

**Stable manifold algorithm:**

\[
x_1(t) = e^{At}x_0, \quad y_1(t) = 0, \quad z_1(t) = 0
\]

\[
\begin{pmatrix}
  x_{k+1} \\
y_{k+1} \\
z_{k+1}
\end{pmatrix}(t, x_0) =
\begin{pmatrix}
e^{At}x_0 + \int_0^t e^{A(t-s)}X(x_k(s), y_k(s), z_k(s)) \, ds \\
- \int_t^\infty e^{B(t-s)}Y(x_k(s), y_k(s), z_k(s)) \, ds \\
- \int_t^\infty e^{C(t-s)}Z(x_k(s), y_k(s), z_k(s)) \, ds
\end{pmatrix}
\]

**Center-stable manifold algorithm:**

\[
x_1(t) = e^{At}x_0, \quad y_1(t) = e^{Bt}y_0, \quad z_1(t) = 0
\]

\[
\begin{pmatrix}
  x_{k+1} \\
y_{k+1} \\
z_{k+1}
\end{pmatrix}(t, x_0, y_0) =
\begin{pmatrix}
e^{At}x_0 + \int_0^t e^{A(t-s)}X(x_k(s), y_k(s), z_k(s)) \, ds \\
e^{Bt}y_0 + \int_0^t e^{B(t-s)}Y(x_k(s), y_k(s), z_k(s)) \, ds \\
- \int_t^\infty e^{C(t-s)}Z(x_k(s), y_k(s), z_k(s)) \, ds
\end{pmatrix}
\]
1. HJE (optimal control problem)

\[ H(x, p) = p^T f(x) - \frac{1}{2} p^T \bar{R}(x)p + \frac{1}{2} x^T Qx = 0 \]

\[ p_1 = \frac{\partial V}{\partial x_1}, \ldots, p_n = \frac{\partial V}{\partial x_n} \]

2. Hamiltonian system

\[ \frac{dx}{dt} = \frac{\partial H}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial x} \]

3. Stable manifold of the Ham. Sys (iterative computation)

4. Optimal feedback control

\[ u = -\bar{R}(x)\frac{\partial V}{\partial x}^T \]
Mechatronics control via optimal control
Acrobot swing up
Acrobot

- Link 1: Passive link
- Link 2: Active link
- Multiple equilibrium
- Control torque
- Free rotation
Acrobot
Acrobot
Acrobot
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"if origin and destination are far enough apart, it will always pay to get on the turnpike and cover distance at the best rate of travel ...“ (Dorfman 1958)
Nonlinear optimal control problems

\[(\Sigma) \quad \dot{x} = f(x) + g(x)u, \quad x(t_0) = x_0\]

\[(OCP_1)_T : \quad \text{Find a control } u \in L^\infty(0, T; \mathbb{R}^m) \text{ such that}\]

\[J(u) = \frac{1}{2} \int_0^T |Cx(t) - z|^2 + |u(t)|^2 \, dt\]

along (\Sigma) is minimized over all \( u \in L^\infty(0, T, \mathbb{R}^m) \).

\[(OCP_2)_T : \quad \text{Find a control } u \in L^\infty(0, T; \mathbb{R}^m) \text{ such that}\]

\[J(u) = \frac{1}{2} \int_0^T x(t)^\top C^\top Cx(t) + |u|^2 \, dt\]

along (\Sigma) is minimized over all \( u \in L^\infty(0, T; \mathbb{R}^m) \) such that \( x(T) = x_1 \).
Turnpike phenomena in OCPs

\[(\Sigma) \quad \dot{x} = f(x) + g(x)u, \quad x(t_0) = x_0\]

\[(\text{OCP}_1)_T : \quad J(u) = \frac{1}{2} \int_0^T |Cx(t) - z|^2 + |u(t)|^2 \, dt\]

\[(\text{OCP}_2)_T : \quad J(u) = \frac{1}{2} \int_0^T x(t)^T C^T Cx(t) + |u|^2 \, dt \quad \text{s.t.} \quad x(T) = x_1\]

Under certain conditions, OCs for above are close to steady-state optimal inputs except at both boundary layers of \([0, T]\).

**Steady-state optimal:**
\[(\text{OCP}_1)_T : (\bar{u}, \bar{x}) \text{ s.t. } \bar{u} = \text{argmin}\{|Cx - z|^2 + |u|^2 \mid f(x) + g(x)u = 0\}\]
\[(\text{OCP}_2)_T : (\bar{u}, \bar{x}) = (0, 0)\]
Turnpike analysis

(Ham) : \[
\begin{aligned}
\dot{x} &= \frac{\partial H}{\partial p} \\
\dot{p} &= -\frac{\partial H}{\partial x}
\end{aligned}
\]

Both problems require to solve 2-point boundary value problem (BVP)

(OCP₁)ₜ : \( x(0) = x₀, \quad p(T) = 0 \)

(OCP₂)ₜ : \( x(0) = x₀, \quad x(T) = x₁. \)

Dynamical system approach provides a geometric framework to analyze BVPs when \( T \gg 1. \)
BVP trajectories --- guess
\[ \dot{x} = f(x), \quad f : \mathbb{R}^n \to \mathbb{R}^n, \quad C^r, \quad \varphi(t, x) : \text{solution} \]

We assume that

\[ f(0) = 0, \quad \lambda(Df(0)) \neq 0, \quad k \text{ in } \mathbb{C}_-, \quad n - k \text{ in } \mathbb{C}_+ \]

- There exist \( C^r \) manifolds \( S \): stable manifold, \( U \): unstable manifold at 0, respectively, defined by

\[
S := \{ x \in \mathbb{R}^n \mid \varphi(t, x) \to 0 \text{ as } t \to \infty \}, \\
U := \{ x \in \mathbb{R}^n \mid \varphi(t, x) \to 0 \text{ as } t \to -\infty \}.
\]

- \( S, U \) are invariant under the flow of \( f \).
- Let \( E^s \): stable subspace (\( \dim = k \)), \( E^u \): unstable subspace (\( \dim = n - k \)) of \( Df(0) \). Then, \( S, U \) are tangent to \( E^s, E^u \), respectively, at \( x = 0 \).
Stable & unstable manifolds
Stable & unstable manifolds & BVP2
Stable & unstable manifolds & BVP2
Stable & unstable manifolds BVP2
How do we justify this??
\[ \dot{x} = f(x); \quad \varphi(t, x) : \text{solution}. \]

Suppose that

\( x = 0 \): hyperbolic equilibrium

\( S \) and \( U \) are \( k, (n - k) \)-dimensional stable and unstable manifolds of \( f \) at 0.

For any \( (n - k) \)-dimensional disc \( B \) in \( U \),

any point \( x \in S \),

any \( (n - k) \)-dimensional disc \( D \) transversal to \( S \) at \( x \) and

any \( \varepsilon > 0 \),

there exists a \( T > 0 \) such that if \( t > T \), \( \varphi(t, D) \) contains an \( (n - k) \)-dimensional disc that is \( \varepsilon \) \( C^1 \)-close to \( B \).
The Lambda lemma (inclination lemma)
Proposition  Suppose that $x_0 \in S$. Then,

- There exists a $T_0 > 0$ such that for every $T > T_0$ there exists a $\rho > 0$ such that 

$$|\varphi(t, y)| \leq Ke^{-\mu t} \text{ for } t \in [0, T], \ y \in B(x_0, \rho),$$

where $B(x_0, \rho)$ is the $n$-dimensional ball centered at $x_0$ with radius $\rho$. Moreover, $\rho \to 0$ when $T \to \infty$.

- There exists a $T_0$ such that for any $T > T_0$ there exist an $(n - k)$-dimensional disc $D$ transversal to $S$ at $x_0$ and a $k$-dimensional disc $E$ transversal to $U$ at $x_1$ such that $\varphi(T, D)$ intersects $\varphi(-T, E)$ at a single point.
Perturbed behavior around (un)stable manifolds
Perturbed behavior around (un)stable manifolds
The turnpike inequality in general systems

Theorem

\[ \dot{x} = f(x); \quad \varphi(t, x) : \text{solution}. \]

Suppose that

\[ x = 0: \text{hyperbolic equilibrium} \]

\[ S \text{ and } U \text{ are } k, (n-k) \text{-dimensional stable and unstable manifolds of } f \text{ at } 0. \]

Then, there exists a \( T_0 > 0 \) such that for every \( T > T_0 \) there exist a \( \rho > 0, \ y_0 \in B(x_0, \rho) \) and \( y_1 \in B(x_1, \rho) \) such that \( \varphi(T, y_0) = y_1 \) and

\[ |\varphi(t, y_0)| \leq K \left[ e^{-\mu t} + e^{-\mu(T-t)} \right] \quad \text{for } t \in [0, T]. \]

Moreover, \( \rho \to 0 \) when \( T \to \infty. \)
The geometric conditions for BVPs

\[(OCP_1)_T : \; x(0) = x_0, \; p(T) = 0 \quad (BVP_1)\]
\[(OCP_2)_T : \; x(0) = x_0, \; x(T) = x_1 \quad (BVP_2)\]

Suppose that \(T \gg 1\). Let \(\pi_1 : (x, p) \mapsto x\) be a natural projection.

\[(OCP_1)_T : \; \text{If } x_0 \in \text{Int}(\pi_1(S)) \text{ and } U \text{ intersects } p = 0 \text{ transversally, then, a solution satisfying } (BVP_1) \text{ exists.}\]

\[(OCP_2)_T : \; \text{If } x_0 \in \text{Int}(\pi_1(S)) \text{ and } x_1 \in \text{Int}(\pi_1(U)), \text{ then, a solution satisfying } (BVP_2) \text{ exists.}\]
Geometric interpretations
Geometric interpretations
Theorem. Suppose that $T \gg 1$.

- If $x_0 \in \text{Int}(\pi_1(S))$ and $x_1 \in \text{Int}(\pi_1(U))$, then a solution satisfying (BVP$_2$) exists.

- If, moreover, $\det D_{x_0} \varphi(t, x_0) \neq 0$ for $t \in [0, T]$, then,
  \[ u^*(t) = -g(x(t, x_0))^\top p(t, x_0) \]
  is the local optimal control.

- Turnpike inequality:
  \[ |u^*(t)| + |x(t, x_0)| \leq K \left[ e^{-\mu t} + e^{-\mu(T-t)} \right] \]

- As $T \to \infty$, $u^*$ converges to the optimal regulation for $x = 0$ from $x_0$ uniformly in $[0, T_1]$, to $u = 0$ uniformly in $[T_1, T_2]$ and to the optimal regulation for $x_1$ from $x = 0$ uniformly in $[T_2, T]$ (in the time-reversed sense).
Corollary. Let $A, B$ are linearization of the system Assume that $T \gg 1$ and $(C, A, B)$ is controllable and detectable, then $(OCP_2)_T$ has a solution for sufficiently small $|x_0|, |x_1|$. 

Key points in the proof:

\[
T_0 S = \{(u, Pu) \mid u \in \mathbb{R}^n\}, \\
T_0 U = \{(u, (PL + I)L^{-1}u) \mid u \in \mathbb{R}^n\},
\]

where $P, L$ are the same as before. But, $L < 0$ holds due to the controllability of $(A, B)$. Clearly, $x_0 \in \text{Int}(\pi_1(S))$ and $x_1 \in \text{Int}(\pi_1(U))$ for sufficiently small $|x_0|, |x_1|$. \(\det D_{x_0, \varphi}(t, x_0) \neq 0\) holds for $t \in [0, T)$ if $|x_0|, |x_1|$ are small enough.
The condition $\det D_{x_0} \varphi(t, x_0) \neq 0$ assures, along the turnpike trajectory, the existence of a local Lagrangian submanifold that is surjectively projectable to $x$-space.

The same condition required in the stable manifold method for optimal regulation.
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- Non-uniqueness of solutions for HJ
  Geometry of stable manifold
Non-unique solution in HJE

1-swing control
J=11.5
$\max |u_{\text{opt}}| \sim 10[V]$

2-swing control
J=5.1
$\max |u_{\text{opt}}| \sim 7[V]$
Non-unique solution in HJE

Analyze the solutions for 1 swing, 2swing, 3swing control

x1-x2-p1 space                                  x1-x2-p2 space

3D figures of the stable manifold including 1~3 swing controllers
Horibe & Sakamoto, IEEE CST, 2018
Using dynamical system theory such as invariant manifolds & lambda-Lemma, a framework to analyze turnpike geometrically has been proposed.

So far, it is shown that we are able to recover some important prior results in simpler manners.

Computation (possibility): method developed for the stable manifold method for optimal regulation.

The existence of Turnpike can be reduced to the existence (& its size) of (un)stable manifolds.

Relation with maximum hands-off control (Nagahara, Quevedo & Nešić, IEEE TAC ‘16) would be very interesting.
Thank you for your attention!