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Continuous Optimal Control Sensitivity Analysis with AD

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ABSTRACT In order to apply a parametric method to a minimum time control problem in celestial mechanics, a sensitivity analysis is performed. The analysis is continuous in the sense that it is done in the infinite dimensional control setting. The resulting sufficient second order condition is evaluated by means of automatic differentiation, while the associated sensitivity derivative is computed by continuous reverse differentiation. The numerical results are given for several examples of orbit transfer, also illustrating the advantages of automatic differentiation over finite differences for the computation of gradients on the discretized problem.

11.1 Introduction

This chapter is concerned with the use of automatic differentiation (AD) in the context of sensitivity analysis of optimal control problems (here, the minimum time transfer of a satellite to a geostationary orbit [107, 404]). Whereas AD is commonly employed on approximated optimal control problems [117], it is seldom used before discretization, in the infinite dimensional setting typical of control. The originality of this article is a use of AD not only to compute gradients of the discretized problem, but also to perform a continuous sensitivity analysis (see also [90] in the case of PDEs). AD then turns to be an efficient way to deal with the cumbersome computations involved in real-life control problems.

The minimum time orbit transfer problem is briefly stated in §11.2. Then, an outline of the specific parametric technique developed to solve it is presented in §11.3; its use requires the sensitivity analysis of interest here, which essentially amounts to integrating a Riccati equation evaluated by AD. The associated sensitivity derivative is computed by reverse differentiation. Some numerical results for the orbit transfer are given in §11.4, especially for very low thrust transfers. Besides, they demonstrate the relevance of AD to evaluate the gradients of the discrete algorithm.

11.2 Low Thrust Orbit Transfer

The problem motivating this study is the minimum time transfer of a satellite towards a geostationary orbit. The dynamics is expressed using the orbital parameters that define the ellipse osculating to the trajectory (since these coordinates are first integrals of the unperturbed motion, they are slowly varying. On the other hand, the expression of the dynamics becomes intricate). We use a more realistic model than in [107], taking into account the variation of the mass m, so that, on a suitable open submanifold of $\mathbb{R}^n$ (n is the dimension of the system; $n = 4$ for the 2D model, $n = 6$ for the 3D one), the dynamics can be written as:

$$\begin{aligned} \dot{x} &= f_0(x) + B(x)u/m \\ \dot{m} &= -\delta|u| \ , \end{aligned}$$

where the control u is the thrust of the engine, and $|.|$ is the Euclidean norm (see [107, 404] for more details). There are also boundary conditions defining the initial and the final orbit,

$$x(0) = x^0, \ m(0) = m^0, \ h(x(t_f)) = 0 \ ,$$

together with a constraint on the maximum modulus of the thrust:

$$|u| \leq F_{max}$$

with F_{max} small (low thrust transfer). The problem of finding an absolutely continuous state (x, m), and an essentially bounded control u that minimize the transfer time t_f will be referred to as $(SP)_{F_{max}}$. Among other results, it is proven in [104] that any optimal control has finitely many switchings so that $|u| = F_{max}$ almost everywhere. As a consequence, $m(t) = m^0 - \delta F_{max} t$ and $(SP)_{F_{max}}$ is reduced to a non-autonomous problem. The technique used to solve it is described in the next section.

11.3 Continuous Sensitivity Analysis

Rather than using direct methods (e.g., direct transcription) that lead to nonlinear programming, we emphasize indirect approaches. They are faster and more accurate for our problem (see [103, 105, 347] for comparisons). Their main drawback is the loss of robustness: the sensitivity of single shooting to the initialization of the adjoint state is well-known [16]. In an attempt to deal with these difficulties, a new parametric technique is introduced in [107] for minimum time control problems. We here give an outline of the method for the problem of §11.2. If we denote by $\phi(\beta)$ the value function of the optimal control problem $(SP)^{\beta}_{F_{max}}$ with fixed final time β (reformulated for convenience on $[0, 1]$ with an obvious change of variables)

$$\begin{aligned}
&1/2|h(x(1))|^2 \to \min \\
&\dot{x} = \beta f(\beta t, x, u) \\
&x(0) = x^0 \\
&|u| \leq F_{max}
\end{aligned}$$

(with $f(t,x,u) = f_0(x) + B(x)u/m(t)$) then the original problem is clearly equivalent to finding the first zero $\bar{\beta}$ of ϕ (which gives a measurement of the non-controllability of the system with respect to the end-point constraint for a prescribed final time β). The advantages of this approach are of three kinds [107]: first, thanks to the separate management of the criterion (that would be treated like any other variable by single shooting), the sensitivity to the initialization of t_f is reduced. Moreover, the ordered search provided by a Newton-like search on ϕ prevents the algorithm from finding too coarse local minima. Finally, since shooting (that will be used on the auxiliary problems, see §11.4) is embedded in this Newton process, the sensitivity to the adjoint state is attenuated too.

Here, though we can prove that ϕ is Lipschitz–and hence almost everywhere differentiable–we need $\mathbb{C}^1$-regularity in order to apply Newton's method to the equation $\phi(\beta) = 0$. To this end, we use the recent sensitivity results for optimal control of [368, 369]. Of course, one could apply a direct method to $(SP)^{\beta}_{F_{max}}$, approximate ϕ by the resulting value function, and use AD to compute the gradients involved in the verification of the usual finite-dimensional sufficient conditions for sensitivity analysis. Again, in order to preserve the continuous information, we prefer to postpone the discretization process and perform the analysis on the continuous form. As in finite dimensions, the idea is to construct an extremal family and to ensure local optimality by sufficient second order conditions that will also be checked by means of AD. For β in $]0, \bar{\beta}[$, if $(x(.,\beta), u(.,\beta))$ is a solution to $(SP)^{\beta}_{F_{max}}$, the Pontryagin maximum principle holds, and there is an absolutely continuous adjoint state $p(.,\beta)$ such that $y = (x,p)$ is a solution of the boundary value problem $(BVP)_\beta$

$$\dot{x} = \partial_p H(t,x,p,u(x,p),\beta) \tag{11.1}$$

$$\dot{p} = -\partial_x H(t,x,p,u(x,p),\beta) \tag{11.2}$$

$$x(0) = x^0,\ p(1) = {}^t h'(x(1))h(x(1)) \tag{11.3}$$

with $H(t,x,p,u,\beta) = \beta(p|f(\beta t,x,u))$ the Hamiltonian and

$$\begin{aligned}
u(.,\beta) &= u(x(.,\beta),p(.,\beta)) \\
&= -F_{max}{}^t B(x(.,\beta))p(.,\beta)/|{}^t B(x(.,\beta))p(.,\beta)|
\end{aligned}$$

whenever ${}^t B(x(.,\beta))p(.,\beta)$ does not vanish. Actually, we assume that

(I1) $u(.,\beta)$ is continuous

Then, if $Z(t,y,\beta)$ denotes the second member of (11.1-11.2), $Z = (Z_1, Z_2) = (\partial_p H(t,x,p,u(x,p),\beta), -\partial_x H(t,x,p,u(x,p),\beta))$, if $\varphi(t,y,\beta)$ is the smooth maximal flow of $\dot{y} = Z(t,y,\beta)$, $(BVP)_\beta$ is equivalent to the shooting equation: find p^0 in $\mathbb{R}^n$ such that

$$S(p^0,\beta) = 0$$

with $S(p^0,\beta) = b(\varphi(1,x^0,p^0,\beta))$ (where $b(y) = p - {}^t h'(x)h(x)$ is the boundary condition of (11.3)). Finally, we need a second regularity condition

(I2) $\partial_p S(p(0,\beta),\beta)$ belongs to $\mathrm{GL}_n(\mathbb{R})$

together with a coercivity condition

(I3) the symmetric Riccati equation below has a bounded solution on $[0,1]$:

$$\dot{Q} = -Q\mathcal{A}(t,\beta) - {}^t\mathcal{A}(t,\beta)Q + Q\mathcal{B}(t,\beta)Q - \mathcal{C}(t,\beta) \tag{11.4}$$

$$((R^f - Q(1))v|v) \geq 0,\ v \in \mathbb{R}^n \tag{11.5}$$

$$\mathcal{A}(t,\beta) = \partial_x Z_1(t,y(t,\beta),\beta),\ \mathcal{B}(t,\beta) = \partial_p Z_1(t,y(t,\beta),\beta)$$

$$\mathcal{C}(t,\beta) = \partial_x Z_2(t,y(t,\beta),\beta)\ ,$$

where R^f is a fixed n by n matrix. Then, we are able to prove that ϕ is $\mathbb{C}^1$ and to give a very simple closed form of its derivative. Indeed, taking advantage of the fact that the constraint on the control is active everywhere (assumption (I1)), the parametric problem $(SP)^\beta_{F_{max}}$ can be rewritten as an abstract optimization problem with equality constraints

$$J(z,\beta) \to \min$$
$$F(z,\beta) = 0$$

with $z = (x,u)$ and obvious expressions for J and F. Then, if we define the Lagrangian $L(z,\lambda,\beta) = J(z,\beta) + \langle \lambda, F(z,\beta)\rangle$ (where $\langle .,. \rangle$ is the duality pairing on the codomain of F), since the dependence $\beta \mapsto (z(\beta),\lambda(\beta))$ is $\mathbb{C}^1$ under the previous assumptions, and since $\partial_z L(z(\beta),\lambda(\beta),\beta) = 0$ (KKT condition), we can compute $\phi'(\beta) = d/d\beta\ J(z(\beta),\beta)$ by reverse differentiation [176] (here on the continuous problem), and get

$$\phi'(\beta) = \partial_\beta L(z(\beta),\lambda(\beta),\beta)\ . \tag{11.6}$$

As a result, we have

Proposition 1 *Under assumptions (I1)-(I3), ϕ is $\mathbb{C}^1$ on $]0,\bar{\beta}[$ and*

$$\phi'(\beta) = H(1,\beta)/\beta\ . \tag{11.7}$$

Proof. We just need to check the assumptions of the sensitivity analysis result of [369]. For a given β_0 in $]0, \bar{\beta}[$, $(SP)^{\beta_0}_{F_{max}}$ has a solution (x_0, u_0) and an adjoint state p_0. The control is smooth by virtue of (I1) and, if $\widetilde{H}(t, x, p, u, \mu, \beta) = \beta(p|f(\beta t, x, u)) + 1/2\mu(|u|^2 - F^2_{max})$ is the augmented Hamiltonian (μ scalar multiplier associated with the inequality constraint $1/2(|u|^2 - F^2_{max}) \leq 0$), one has $\nabla_u \widetilde{H}(t, x_0, p_0, u_0, \mu_0, \beta_0) = 0$ by taking $\mu_0 = \beta_0 |{}^t B(x_0) p_0| / (m(t) F_{max}) \geq 0$. Accordingly, μ_0 is smooth and (I1) implies that strict complementarity holds ($\mu_0 > 0$ on $[0, 1]$). Moreover, $\nabla^2_{uu} \widetilde{H}(t, x_0, p_0, u_0, \mu_0, \beta_0) = \mu_0 \mathbf{I}$ (**I** identity matrix) in order that the strict Legendre-Clebsch condition is fulfilled. Then, with (I2) and (I3), all the assumptions of [369] are valid so that, for any β in an open neighbourhood of β_0,

$$\phi'(\beta) = \int_0^1 \partial_\beta H(t, \beta) dt$$

by virtue of (11.6) (lemma 1 of [107]). Now, along the optimal trajectory,

$$d/dt(tH) = H + t\dot{H} = H + t\partial_t H = \beta \partial_\beta H \ ,$$

so $\phi'(\beta) = H(1, \beta)/\beta$, which concludes the proof. □

Both conditions (I2) and (I3) are only verifiable numerically: (I2) is simply the regularity of the Jacobian of the shooting function (checked when solving the auxiliary problem $(SP)^{\beta}_{F_{max}}$ by shooting). The Ricatti equation (11.4-11.5) requires the computation of $3n^2$ partial derivatives and is assembled using AD as explained in the next section.

11.4 Numerical Results

The numerical computation is done in two steps. First, for a given thrust, $(SP)_{F_{max}}$ is solved by the parametric approach of §11.3; $\phi(\beta)$ is evaluated by shooting on the auxiliary problem $(SP)^{\beta}_{F_{max}}$ (which allows the numerical verification of (I2)); and ϕ' is evaluated using either (11.7), which is extremely easy to compute, or a finite differences approximation, depending on the precision of the shooting resolution. Then, the Riccati equation (I3) is integrated to check coercivity, backwards since it is straightforward to find a matrix $Q(1)$ matching the boundary condition (11.5), and since $y(1, \beta)$ is provided by shooting. AD is used to generate the gradients. Indeed, we deal with a large system, so there are many derivatives to determine ($3n^2$, that is 48 in 2D, and 108 in the 3D case). Moreover, computing the required second order derivatives of the dynamics is a cumbersome process because of the choice of coordinates (one has to deal with trigonometric rational fractions). In the 3D case for instance, even the first order derivatives of the dynamics for the adjoint equation are evaluated by AD. As the Fortran code for the boundary value problem was available, ADIFOR2.0 [79] was a natural choice. AD is also used in a more classical but yet efficient fashion to find the exact Jacobian of the shooting function, $\partial_p S(p^0, \beta)$, by differentiating the numerical integrator (Runge-Kutta of order 4). A comparison with finite differences (FD) is provided Table 11.1 where the transfer times for various thrusts are detailed in the 2D case (compare [347, 208, 404, 103]). The results we obtained using AD are systematically more accurate. Two examples of optimal trajectories and verification of the coercivity condition are given in Figures 11.1 and 11.2.

TABLE 11.1. Minimum transfer times

	t_f		$\phi(t_f)$		Execution	
F_{max}	FD	AD	FD	AD	FD	AD
60	14.732	14.732	5e-28	7e-29	12	14
24	34.133	34.133	2e-22	3e-27	25	25
12	69.294	69.294	2e-25	2e-21	60	40
9	93.187	91.930	3e-19	1e-26	54	70
6	141.64	139.37	3e-13	2e-17	122	86
3	278.98	278.98	1e-24	1e-27	285	217
2	420.10	420.10	1e-17	1e-26	257	485
1.4	597.92	598.12	4e-18	5e-13	485	648
1	839.97	836.86	5e-12	3e-13	496	504
0.7	1195.7	1195.7	2e-12	9e-15	1084	1106
0.5	1685.2	1674.8	3e-12	2e-12	1978	1391
0.3	2838.4	2797.7	7e-10	4e-13	2128	1938

Thrusts are in Newtons, transfer times in hours, and execution times (on an HP PA-C160) in seconds.

11.5 Conclusion

Automatic differentiation has been used in two ways: on the discretized problem to evaluate the Jacobian of the shooting function, but also on the original control problem to check a continuous sensitivity condition (and then to find a continuous exact gradient by reverse differentiation). In the first case, AD provides a much more accurate computation than finite differences. In the second, it is the most practical way to assemble the Riccati equation connected to the coercivity condition. For the real-life control problem considered, the large dimension of the dynamics together with its complexity make any hand-made computation cumbersome, if not impracticable. A similar analysis with respect to the parameter F_{max} is currently worked out [102].

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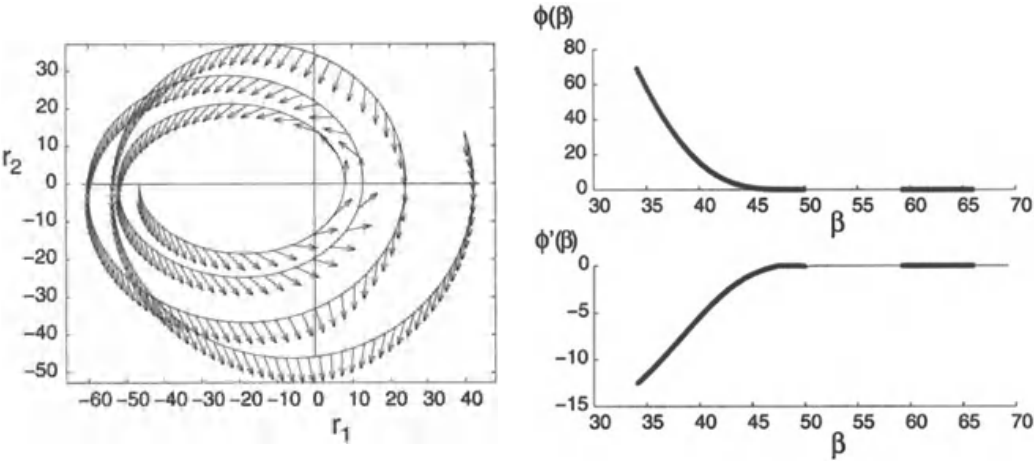

FIGURE 11.1. Thrust of 12 Newtons (3 day transfer). Left, the optimal trajectory (the arrows represent the control). Right, the evaluation of ϕ and ϕ'. Points where the Riccati equation has been successfully integrated are marked with a $*$.

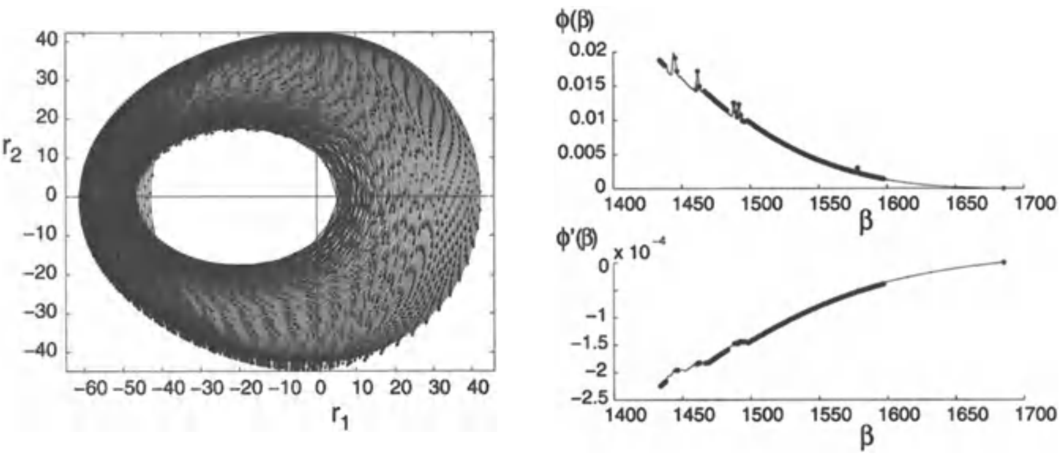

FIGURE 11.2. Thrust of 0.5 Newton (2 month transfer). The result is typical of the low thrust case: the coercivity condition is more difficult to check in the neighbourhood of the solution, and jumps are observed on the function, due to local minima.