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Energy minimization of single input orbit transfer by averaging and continuation

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Abstract

This article deals with the transfer between Keplerian coplanar orbits using low propulsion. We focus on the energy minimization problem and compute the averaged system, proving integrability and relating the corresponding trajectories to a three-dimensional Riemannian problem that is analyzed in details. The geodesics provide approximations of the extremals of the energy minimization problem and can be used in order to evaluate the optimal trajectories of the time optimal and the minimization of the consumption problems with continuation methods. In particular, minimizing trajectories for transfer towards the geostationary orbit can be approximated in suitable coordinates by straight lines.
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Résumé

On s'intéresse dans cet article au transfert entre des orbites Kepleriennes en propulsion faible. On considère le problème de minimisation de l'énergie et on calcule le système moyenné. On prouve que les trajectoires correspondantes sont les extrémales d'un problème Riemannien en dimension trois et qu'elles sont intégrables. Ces trajectoires sont des approximations des extrémales du problème de minimisation d'énergie et peuvent être utilisées pour calculer les trajectoires optimales du problème de temps minimal ou de maximisation de la masse finale par des méthodes de continuation. En particulier pour le problème de transfert vers l'orbite géostationnaire, les trajectoires minimisantes peuvent être approchées par des droites dans des coordonnées adaptées.
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Keywords: Orbital transfer with low thrust; Energy minimization; Averaging; Continuation methods

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Mots-clés : Transfert orbital à poussée faible ; Minimisation de l'énergie ; Moyennation ; Méthodes de continuation

1. Introduction

Current research projects concern orbital transfer of a satellite between elliptic Keplerian orbits with electro-ionic propulsion where the performance of the engine is high but the thrust is very low [1]. Related optimal control problems consist in minimizing the transfer time or the consumption of the engine. Both problems can be written

$$\int_0^T f^0(x,u)\,dt \to \min$$

for the trajectories of $\dot{x} = f(x,u)$, where the function of time u is the thrust and $|u| \leqslant \varepsilon$. From the maximum principle [2], optimal trajectories are to be found among a set of *extremals* (x, p, u) that are solutions of the equations

$$\dot{x} = \frac{\partial H}{\partial p}(x,p,u), \qquad \dot{p} = -\frac{\partial H}{\partial x}(x,p,u)$$

while the control u satisfies the condition

$$H(x,p,u) = \max_{|v| \leqslant \varepsilon} H(x,p,v),$$

where $H(x,p,u) = p^0 f^0(x,u) + \langle p, f(x,u)\rangle$ is the Hamiltonian, $p^0 \leqslant 0$ is a constant and p is the adjoint vector. In orbital transfer, the extremal system is highly complicated and is analyzed mainly using numerical simulations, see [3] for the time optimal case and [4] for the minimization of the consumption. With this approach based on the maximum principle, the optimal control problem is solved using a *shooting method* and the adjoint initial vector p_0 is solution of a nonlinear shooting equation. Therefore, it is crucial to have a good initial guess on p_0. Such an approximation can be obtained using *continuation methods* on the maximal thrust ε (see [3]) or by considering a deformation of the cost. For instance, we can use a path connecting the energy to the consumption [4]. In this framework, an important question is to determine if the extremal solutions can be approximated by trajectories of an integrable system.

In this article, we analyze the energy minimization problem for low thrust using the averaged system. For the sake of simplicity, we shall restrict our analysis to coplanar transfers. Moreover, we shall only consider the single-input case where the control is oriented along the tangential direction. This corresponds to practical constraints [1] and is related to the standard analysis of the effect of the drag term in space mechanics [5]. Our main contribution is to show that the averaged system is integrable and is connected to a Riemannian problem whose distance is an approximation of the original minimization problem, see [6] and [7] for related works.

The organization of this article is the following. In Section 2, we recall the controlled Kepler equation and introduce the standard geometric coordinates, namely orbit elements and longitude, associated respectively with *slow* and *fast* variables when averaging for low thrust. We define the energy minimization problem and the averaged problem. The averaged system is then related to a three-dimensional Riemannian problem whose geodesics and length provide an approximation up to first order in ε of the extremal solutions, and to second order of the value function of the original problem [6,8]. Section 3 is the main contribution of this article. Explicit computations of the averaged system are given. The resulting three-dimensional Riemannian problem is analyzed in details. Normal coordinates are computed that allow to integrate the geodesics curves and to study their optimality for the associated length minimization problem. In particular, the averaged trajectories in the case of a transfer towards a geostationary orbit are straight lines. Section 4 contains preliminary computations. The Riemannian spheres are presented and we are able to conclude about global optimality by inspecting their smoothness. The continuation on the shooting method is illustrated taking a path between the averaged system and the real one.

2. Preliminaries

2.1. *Controlled Kepler equation*

In Cartesian coordinates, the equation describing the coplanar orbital transfer is

$$\ddot{q} = -\mu \frac{q}{|q|^3} + \frac{F}{m},$$

where q is the position measured in a plane identified with the equatorial plane, F is the thrust bounded by $|F| \leqslant \varepsilon$ and m is the mass, its evolution being described by

$$\dot{m} = -\frac{|F|}{v_e}.$$

Here before, v_e is the constant gas ejection speed. We decompose the thrust in the *tangential-normal* frame:

$$F = u_1 F_1 + u_2 F_2 \quad \text{where } F_1 = \frac{\dot{q}}{|\dot{q}|} \frac{\partial}{\partial \dot{q}}.$$

The relevant optimal control problems are to minimize the time or the mass consumption. The latter amounts to an L^1-minimization problem and we introduce the energy minimization problem, replacing so the L^1-norm by the L^2-norm: $\int_0^T |u(t)|^2\, dt$. In this article, we shall restrict our analysis to the constant mass case (neglecting the mass variation). Besides, considering the energy minimization problem makes the computation of the averaged system analytically tractable.

In contrast with [8] where the multiple input case is considered, we assume that the thrust is tangential, so that $F = uF_1$, where u is a scalar function, $|u| \leqslant \varepsilon$. It is moreover crucial to represent the system in the adapted coordinates of [6] denoted (v, x) where $x = (n, e, \omega)$ belongs to the *elliptic domain*

$$X = \{n > 0,\ 0 < e < 1,\ \omega \in \mathbf{S}^1\}.$$

Here before, we have used the *elliptic elements* where n is the *mean movement* equal to $\sqrt{\mu/a^3}$ where a is the semi-major axis, e is the *eccentricity*, ω is the *argument of the pericenter*, and v is the *true longitude* corresponding to the angle of the pericenter: $v = l - \omega$ where l is the polar angle or *longitude*. The equations are:

$$\dot{n} = -\frac{3n^{2/3}}{\mu^{1/3}} \left(\frac{1 + 2e\cos v + e^2}{1 - e^2} \right)^{1/2} u, \tag{1}$$

$$\dot{e} = \frac{2(e + \cos v)}{(\mu n)^{1/3}} \left(\frac{1 - e^2}{1 + 2e\cos v + e^2} \right)^{1/2} u, \tag{2}$$

$$\dot{\omega} = \frac{2\sin v}{(\mu n)^{1/3} e} \left(\frac{1 - e^2}{1 + 2e\cos v + e^2} \right)^{1/2} u, \tag{3}$$

$$\dot{l} = n\frac{(1+e\cos v)^2}{(1-e^2)^{3/2}}. \tag{4}$$

The coordinates are singular for circular orbits but the singularity $e=0$ can be removed by using the *eccentricity vector*:

$$e_x = e\cos\omega, \qquad e_y = e\sin\omega.$$

The control system is of the form:

$$\dot{l} = \omega_0(l,x), \qquad \dot{x} = uF(l,x),$$

where $x \in X$, $l \in \mathbf{S}^1$, F is a smooth vector field on $\mathbf{S}^1 \times X$ and ω_0 is a smooth positive function defined on $\mathbf{S}^1 \times X$.

2.2. Minimum energy control and averaging

The energy can be written

$$\int_0^T u^2\,dt = \int_{l_0}^{l_f} \frac{u^2\,dl}{\omega_0(l,x)}$$

so that, after replacing time by the cumulated longitude l, the Hamiltonian to consider from the maximum principle is

$$H(l,x,p,u) = \frac{p^0u^2 + uP(l,x,p)}{\omega_0(l,x)}$$

and P is the Hamiltonian lift $\langle p, F(l,x)\rangle$. Up to a rescaling, we can assume $\mu = 1$.

In the framework of our application, we are interested in the action of small controllers so that, replacing u by εu where $|u| \leqslant 1$, the Hamiltonian is rescaled as

$$H(l,x,p,u) = \varepsilon\frac{-u^2/2 + uP(l,x,p)}{\omega_0(l,x)}$$

where p^0 has been normalized to $-1/(2\varepsilon)$ in the *normal case* ($p^0 \neq 0$).

In order to make explicit the computation of the extremal controllers, we drop the bound $|u| \leqslant 1$ (practically, the constraint $|u| \leqslant 1$ will be fulfilled for large enough transfer times). The maximization condition leads then to $\partial H/\partial u = 0$ and extremal controls are $u = P(l,x,p)$. Plugging such controls into H, we obtain the *true Hamiltonian*

$$H(l,x,p) = \varepsilon\frac{P^2(l,x,p)}{2\omega_0(l,x)}. \tag{5}$$

We drop the parameter ε, which amounts to parameterizing by $\tilde{l} = \varepsilon l$ instead of l. Since H is 2π-periodic in the angular variable, we introduce

Definition 1. The averaged Hamiltonian is

$$\overline{H}(x,p) = \frac{1}{2\pi}\int_0^{2\pi} H(l,x,p)\,dl.$$

From [6] which uses standard approximation results between trajectories of H and $\overline{H}$, see [7,9], the following is true.

Proposition 2. *Let $z(l)$ and $\bar{z}(l)$ be respective integral curves of H and $\overline{H}$ with same initial conditions, then the difference $z - \bar{z}$ is uniformly of order* $\mathrm{o}(\varepsilon)$ *for a length of order* $1/\varepsilon$*, and the difference between the respective energy costs is of order* $\mathrm{o}(\varepsilon^2)$.

An important conceptual step introduced in [8] is to relate $\overline{H}$ to an optimal control problem. We need the following, see [10] for details.

2.3. Sub-Riemannian problems

Let $F_1, \ldots, F_m$ be smooth vector fields on X and assume that the distribution $\mathcal{D} = \mathrm{Span}\{F_1, \ldots, F_m\}$ is of constant rank m. A *sub-Riemannian problem* (or *SR-problem*) is an optimal control problem of the form

$$\int_0^T \left(\sum_{i=1}^m u_i^2(t)\right)^{1/2} dt \to \min$$

$$\dot{x} = \sum_{i=1}^m u_i F_i(x).$$

Geometrically, this amounts to minimizing the length of a curve tangent to $\mathcal{D}$. If $m = n = \dim X$, this is a standard *Riemannian problem*.

According to Maupertuis principle, the problem is equivalent to minimizing the *energy* of the curve, $\int_0^T |u|^2(t)$, where the transfer time T is fixed. From the maximum principle, optimal trajectories are solutions of

$$\dot{x} = \frac{\partial H}{\partial p}, \qquad \dot{p} = -\frac{\partial H}{\partial x}, \qquad \frac{\partial H}{\partial u} = 0$$

where

$$H(x, p, u) = p^0 \sum_{i=1}^m u_i^2 + \sum_{i=1}^m u_i P_i$$

with $P_i = \langle p, F_i \rangle$ for $i = 1, \ldots, m$ and $p^0 \leqslant 0$ is a constant which is normalized either to 0 or -1. In the second or *normal* case, extremal controls verify $u_i = P_i$ and normal extremals are trajectories of the Hamiltonian equation associated with the true Hamiltonian

$$H(x, p) = \frac{1}{2} \sum_{i=1}^m P_i^2.$$

Geometrically, this Hamiltonian defines a Riemannian metric on the distribution $\mathcal{D}$, where by construction, the vector fields $\{F_1, \ldots, F_m\}$ form an orthonormal frame.

2.4. SR-problem of the averaged system

Definition 3. The averaged Hamiltonian $\overline{H}$ is a nonnegative quadratic form $\overline{w}(x)$ in p and the averaged system is said to be *regular* if this form has constant rank.

Proposition 4. *If the averaged system is regular of rank k, the averaged Hamiltonian $\overline{H}$ can be written as a sum of k squares,*

$$\overline{H}(x,p)=\frac{1}{2}\sum_{i=1}^{k}\overline{P}_i^2$$

with $\overline{P}_i=\langle p,\overline{F}_i(x)\rangle$, and $\overline{H}$ is the Hamiltonian associated with the SR-problem:

$$\int_0^T\sum_{i=1}^{k}u_i^2(t)\,dt\to\min,$$

$$\dot{x}=\sum_{i=1}^{k}u_i\overline{F}_i(x).$$

If $k=n=\dim X$, then $\overline{H}$ is the Hamiltonian of a Riemannian problem on X.

Proof. If the problem is regular of rank k, then there exists an orthogonal matrix $R(x)$ such that if $P=R(x)p$ then $\overline{w}(x)$ is written as a sum of squares:

$$\overline{w}(x)=\frac{1}{2}\sum_{i=1}^{k}\lambda_i(x)\overline{P}_i^2(x)$$

where $\lambda_1,\dots,\lambda_k$ are positive functions. Hence we can write

$$\overline{w}(x)=\frac{1}{2}\sum_{i=1}^{k}\bigl(\lambda_i^{1/2}(x)\overline{P}_i(x)\bigr)^2=\frac{1}{2}\sum_{i=1}^{k}\bigl\langle p,\overline{F}_i(x)\bigr\rangle^2$$

defining so k smooth vector fields $\overline{F}_1,\dots,\overline{F}_k$ on X. $\square$

3. Averaged Hamiltonian of energy minimization coplanar transfer

3.1. Computations

According to Eqs. (1)–(4), the true Hamiltonian $H(l,x,p)$ can be written $(1/2)\langle A(v,x)p,p\rangle$ where $A(v,x)$ is the symmetric matrix with coefficients

$$A_{nn}(v,x)=9n^{1/3}(1-e^2)^{1/2}\frac{1+2e\cos v+e^2}{(1+e\cos v)^2},\tag{6}$$

$$A_{ee}(v,x)=\frac{4(1-e^2)^{5/2}}{n^{5/3}}\frac{1}{1+2e\cos v+e^2}\left(\frac{e+\cos v}{1+e\cos v}\right)^2,\tag{7}$$

$$A_{\omega\omega}(v,x)=\frac{4(1-e^2)^{5/2}}{n^{5/3}e^2}\frac{1}{1+2e\cos v+e^2}\left(\frac{\sin v}{1+e\cos v}\right)^2,\tag{8}$$

$$A_{ne}(v,x)=-\frac{6(1-e^2)^{3/2}}{n^{2/3}}\frac{e+\cos v}{(1+e\cos v)^2},\tag{9}$$

$$A_{n\omega}(v,x)=-\frac{6(1-e^2)^{3/2}}{n^{2/3}e}\frac{\sin v}{(1+e\cos v)^2},\tag{10}$$

$$A_{e\omega}(v,x)=\frac{4(1-e^2)^{5/2}}{n^{5/3}e}\frac{1}{1+2e\cos v+e^2}\frac{(e+\cos v)\sin v}{(1+e\cos v)^2}. \tag{11}$$

The averaged Hamiltonian is similarly written $(1/2)\langle\overline{A}(x)p,p\rangle$ where $\overline{A}(x)$ is the symmetric matrix whose elements are the averaged of the six coefficients (6)–(11) previously defined. The computation is lengthy but straightforward if we use the following obvious remarks.

Remark 5. We can replace in the integration the variable l by the true longitude:

$$\overline{H}(x,p)=\frac{1}{2\pi}\int_0^{2\pi}H(v,x,p)\,dv.$$

Remark 6. Each integrand is of the form $Q(\cos v,\sin v)$ where Q is a rational fraction. Such integrals can be computed using the residue theorem by setting $z=e^{iv}$:

$$\int_0^{2\pi}Q(\cos v,\sin v)\,dv=\int_{\mathbf{S}^1}f(z)\,dz=2i\pi\sum_k \mathrm{Res}(f,z_k)$$

where

$$f(z)=\frac{1}{iz}Q\left(\frac{z+z^{-1}}{2},\frac{z-z^{-1}}{2i}\right)$$

and the z_k's are the poles of f contained in the interior of the unit circle. From our expressions, the poles are related to the roots associated with $1+e\cos v$, $1+2e\cos v+e^2$, i.e. either the roots of $ez^2+2z+e=e(z-z_1)(z-z_2)$ with $z_1=(-1-\sqrt{1-e^2})/e$, $z_2=(-1+\sqrt{1-e^2})/e$ and $|z_2|<1$ is the only root in the unit disk, or the roots of $ez^2+(1+e^2)z+e=e(z+1/e)(z+e)$ and $-e$ is the only root in the unit disk (the case $e=0$, corresponding to circular orbits being excluded and derived as a limit case).

Proposition 7. *The matrix* $\overline{A}(x)$ *is diagonal*:

$$\overline{A}_{nn}(x)=9n^{1/3},$$
$$\overline{A}_{ee}(x)=\frac{1}{n^{5/3}}\frac{4(1-e^2)^{3/2}}{1+\sqrt{1-e^2}},$$
$$\overline{A}_{\omega\omega}(x)=\frac{1}{n^{5/3}}\frac{4(1-e^2)}{e^2(1+\sqrt{1-e^2})}$$

and $\overline{A}_{ne}(x)=\overline{A}_{n\omega}(x)=\overline{A}_{e\omega}(x)=0.$

The averaged Hamiltonian is thus

$$\overline{H}=\frac{1}{2n^{5/3}}\left[9n^2p_n^2+\frac{4(1-e^2)^{3/2}}{1+\sqrt{1-e^2}}p_e^2+\frac{4(1-e^2)}{1+\sqrt{1-e^2}}\frac{p_\omega^2}{e^2}\right]$$

and, writing $\overline{H}=(1/2)\sum_{i=1}^3\langle p,\overline{F}_i(x)\rangle^2$, we get:

$$\overline{F}_1 = \frac{3n}{n^{5/6}} \frac{\partial}{\partial n},$$
$$\overline{F}_2 = \frac{2}{n^{5/6}} \frac{(1-e^2)^{3/4}}{(1+\sqrt{1-e^2})^{1/2}} \frac{\partial}{\partial e},$$
$$\overline{F}_3 = \frac{2}{n^{5/6}} \frac{(1-e^2)^{1/2}}{e(1+\sqrt{1-e^2})^{1/2}} \frac{\partial}{\partial \omega}.$$

The following holds.

Theorem 8. *The averaged Hamiltonian $\overline{H}$ is associated with the three-dimensional metric*

$$\bar{g} = \frac{dn^2}{9n^{1/3}} + n^{5/3} \frac{1+\sqrt{1-e^2}}{4(1-e^2)^{3/2}}\, de^2 + n^{5/3} \frac{(1+\sqrt{1-e^2})e^2}{4(1-e^2)}\, d\omega^2 \tag{12}$$

and (n, e, ω) are orthogonal coordinates, singular for circular orbits $(e = 0)$.

Remark 9. The averaged problem is Riemannian, although the initial problem is a *sub-Riemannian problem with drift* [1] and only one control in dimension four, see Section 2. Actually, it is shown in [1] that brackets up to length two ensure controllability of the non-averaged system. Such brackets are generated by the averaging process (compare with the multi-input case [11] where only brackets of length one are required).

3.2. Normal coordinates

3.2.1. Geometric preliminaries

The elliptic elements (n, e, ω) are *orthogonal coordinates* [12], which is an important geometric reduction for the metric. Further normalizations are needed to describe the geometric properties of the extremals and perform a complete analysis. In particular, since the Hamiltonian is not depending on ω, the coordinate is *cyclic* and its dual variable p_ω is a first integral of the averaged motion. As a result, if we restrict the system to the four-dimensional symplectic subspace $\{\omega = p_\omega = 0\}$, the Hamiltonian is analytic and is associated with a planar Riemannian metric defined by

$$ds^2 = \frac{dn^2}{9n^{1/3}} + n^{5/3} \frac{1+\sqrt{1-e^2}}{4(1-e^2)^{3/2}}\, de^2. \tag{13}$$

Geometrically, the condition $p_\omega = 0$ is the transversality condition for a transfer towards a circular orbit, where the angle of the pericenter is not prescribed. This is the case for the important practical problem of steering the system to the *geostationary orbit*.

The main step when computing a normal form is to reduce the corresponding metric.

Proposition 10. *In the appropriate domain, the metric $\bar{g}$ is isometric to $ds^2 = du^2 + u^2(dv^2 + G(v)\, dw^2)$.*

Proof. Consider the two-dimensional restriction of the metric (13),

$$ds^2 = \frac{dn^2}{9n^{1/3}} + n^{5/3} \frac{1+\sqrt{1-e^2}}{4(1-e^2)^{3/2}}\, de^2.$$

Setting $u = (2/5)n^{5/6}$ it becomes $du^2 + u^2\,dv^2$ where v is defined by

$$dv^2 = \left(\frac{5}{4}\right)^2 \frac{1+\sqrt{1-e^2}}{(1-e^2)^{3/2}}\,de^2$$

and a straightforward integration gives us:

$$v = \frac{5}{4}\arcsin\bigl(1 - 2\sqrt{1-e^2}\,\bigr). \tag{14}$$

The change of coordinates (14) is well defined either on $]-1,0[$ or $]0,1[$, and valued in both cases in the interval $]-\pi/(2c), \pi/(2c)[$ with $c = 4/5$. The effect of the singularity $e = 0$ is that the inverse mapping of (14) is the multiform function

$$e = \pm\left[1 - \left(\frac{1-\sin(cv)}{2}\right)^2\right]^{1/2}. \tag{15}$$

If we set $w = \omega$ and

$$G(v) = \frac{(1+\sqrt{1-e^2}\,)e^2}{4(1-e^2)}$$

the normal form is computed. □

An important corollary for applications is the following.

Corollary 11. *In suitable coordinates, the geodesics associated with the averaged transfer towards circular orbits are straight lines.*

Proof. In accordance with the previous computation, the metric is isometric to the *polar metric* $du^2 + u^2\,dv^2$, where $u = (2/5)n^{5/6}$ is positive in the domain. If we set $x = u\cos v$ and $y = u\sin v$, the polar metric takes the form of the flat metric $dx^2 + dy^2$ and the geodesic curves are straight lines. □

Remark 12. The two-dimensional elliptic subdomain, defined in polar coordinates by the two copies of $\{u > 0,\ v \in]-\pi/(2c), \pi/(2c)[\}$ which have to be glued together along $v = -\pi/(2c)$, is not convex since $c = 4/5 < 1$, that is not *geodesically convex*, geodesics being straight lines in $x = u\cos v$, $y = u\sin v$ coordinates. This fact is related to existence issues, see [11] for the discussion in the multi-input case.

3.3. Integrability of the averaged system

Theorem 13. *The extremal flow defined by the averaged Hamiltonian $\overline{H}$ is completely integrable.*

Proof. Explicit expressions can be obtained for the trajectories of $\overline{H}$ in the original elliptic elements, but a shortest proof is to use the normal form $g = du^2 + u^2(dv^2 + G(v)\,dw^2)$. First of all, consider the metric

$$ds^2 = dv^2 + G(v)\,dw^2. \tag{16}$$

The function $G(v)$ is related to the Gauss curvature by:

$$K = -\frac{1}{\sqrt{G}}\frac{\partial^2\sqrt{G}}{\partial v^2}. \tag{17}$$

The metric can be written

$$ds^2 = G(v)\left[\left(\frac{dv}{\sqrt{G(v)}}\right)^2 + dw^2\right]$$

and belongs to the class of *Liouville metrics* $(f(x) + g(y))(dx^2 + dy^2)$ where f and g are smooth positive functions. A standard result of Riemannian geometry in two dimensions asserts that the extremal flow of a Liouville metric is completely integrable [12]. This is straightforward in our case, since p_w is a linear first integral and we have two independent and commuting such integrals: $\overline{H}'$ and p_w, where $\overline{H}'$ is the Hamiltonian

$$\overline{H}' = \frac{1}{2}\left(p_v^2 + \frac{p_w^2}{G(v)}\right).$$

Consider now the averaged Hamiltonian $\overline{H}$ which can be written in our normal coordinates:

$$\overline{H} = \frac{1}{2}p_u^2 + \frac{1}{u^2}\overline{H}'.$$

We have:

$$\dot{u} = \frac{\partial \overline{H}}{\partial p_u} = p_u,$$

$$\dot{p_u} = -\frac{\partial \overline{H}}{\partial u} = \frac{1}{u^3}\left(p_v^2 + \frac{p_w^2}{G(v)}\right).$$

Then, if $V = up_u$, $\dot{V} = 2\overline{H} = C_1$, since $\overline{H}$ is a first integral. Hence, if $s = u^2$, we have $\dot{s} = 2u\dot{u} = 2V$ and $\ddot{s} = 2C_1$. Therefore $s(t)$ is a polynomial of degree 2,

$$s(t) = C_1 t^2 + \dot{s}(0)t + s(0)$$

with $s(0) = u^2(0)$ and $\dot{s}(0) = 2u(0)p_u(0)$, and $p_u = \dot{u}$.

The remaining equations can be integrated since $\overline{H}'$ is the Hamiltonian of the Liouville metric $dv^2 + G(v)\,dw^2$ defined previously, and because we can use the reparameterization $dT = dt/u^2$ where $s = u^2$ is known. □

4. Optimality results and Riemannian spheres

4.1. Preliminaries

Consider the averaged Hamiltonian $\overline{H}(x, p)$ where $x = (n, e, \omega)$ and $p = (p_n, p_e, p_\omega)$ associated with the metric $\overline{g}$ defined by (12). We parameterize geodesics by arc-length by restricting the averaged Hamiltonian to the level set $\overline{H} = 1/2$. We denote by $z(t, z_0)$ an extremal curve $z(t, z_0) = (x(t, z_0), p(t, z_0))$ originating from $z_0 = (x_0, p_0)$. The *exponential mapping* is the map

$$\exp_{x_0} : (t, p_0) \mapsto x(t, z_0)$$

where x_0 is fixed. We note $S(x_0, r)$ the Riemannian sphere with center x_0 and radius r. A *conjugate time* t_c is a time for which the exponential mapping is not an immersion. The corresponding point is called a *conjugate point*. The *conjugate locus* $C(x_0)$ is the set of first conjugate points when we consider all the extremals starting from x_0. The point where the extremal ceases to be minimizing is called the *cut point* and the set of cut points form the *cut locus* $L(x_0)$.

Standard results of Riemannian geometry [13] are applied to make a complete analysis. If the radius is small enough, the sphere is formed by extremities of extremal curves and we get global results by extending such curves. A cut point is either a conjugate point or a point where two minimizing geodesics with equal length are intersecting. At such points, the sphere is not smooth. As a consequence, the inspection of the extremal flow permits to decide on global optimality.

4.2. Geometric analysis and global optimality

Using the normal coordinates, the metric $\bar{g}$ becomes $du^2 + u^2(dv^2 + G(v)\,d\omega^2)$ where $u = (2/5)n^{5/6}$ and $v = (5/4)\arcsin(1 - 2\sqrt{1-e^2}\,)$. The polar metric $du^2 + u^2\,dv^2$ is the flat metric if we set $x = u\cos v$, $y = u\sin v$, and the extremal curves are globally straight lines. This allows to solve the problem of transfer towards circular orbits.

To complete the analysis, it is sufficient to analyze the extremals of the two-dimensional Riemannian metric $dv^2 + G(v)\,dw^2$. The covariant function $G(v)$ is related to the Gauss curvature by (17). It governs the distribution of conjugate points according to Jacobi equation, and the conjugate locus can be computed.

4.3. Numerical simulations

Although explicit computations are tractable thanks to complete integrability, we can also use numerical simulations to represent Riemannian spheres and conclude about optimality. Besides, those simulations are necessary to give comparisons between the extremals of the averaged and the original Hamiltonians. The method of continuation is then fruitful to initialize the computation of the real system trajectories.

In all figures, we consider $x_0 = (e_0, n_0, \omega_0) = (0.75, 0.5, 0)$. On Fig. 1 we represent geodesics of the transfer to circular orbits, that is minimizing extremals such that $\omega = p_\omega = 0$. Fig. 2 is a projection of the extremals in the plane (v, w) and corresponds to extremals of the metric defined in (16). On Fig. 3 we eventually compare a minimizing trajectory of the averaged system with the

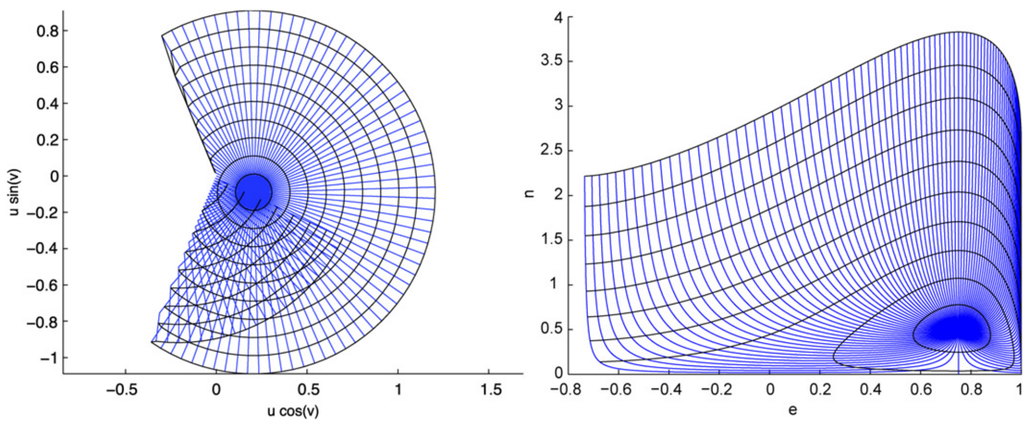

Fig. 1. Geodesics of the transfer towards circular orbits up to length 1, and spheres for radii between $1e-1$ and 1. On the left graph, flat coordinates are used and the multiform character of the change of variables (15) is illustrated by the reflexion phenomenon on $v = -5\pi/8$. As shown on the right graph in coordinates (e, n), there is no self-intersection in the two-dimensional elliptic subdomain, and the singularity $e = 0$ is smoothly crossed by geodesics.

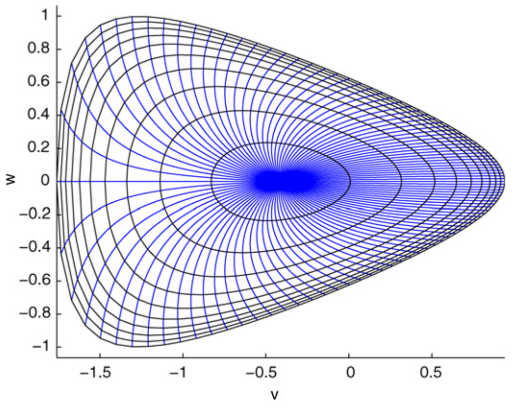

Fig. 2. Geodesics up to length 1 of the transfer projected on the (v, w)-space, and associated spheres for radii between $1e-1$ and 1.

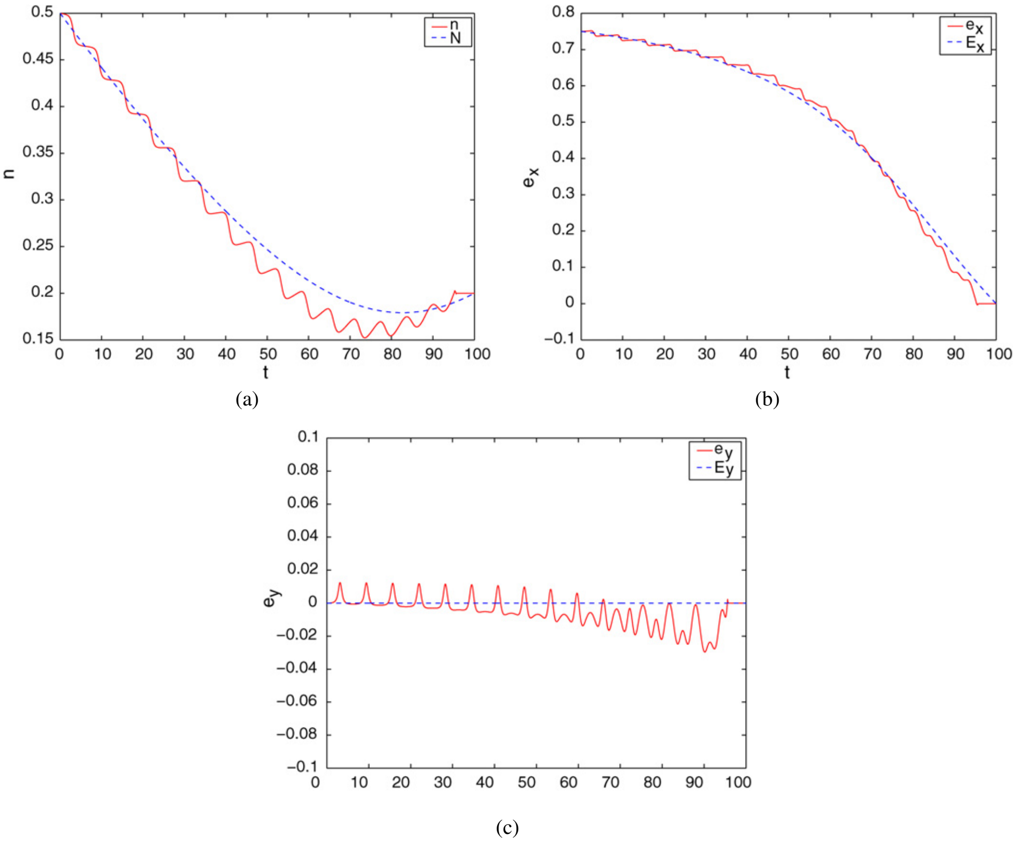

Fig. 3. Computation by continuation of the non-averaged solution. The averaged trajectories ($\varepsilon = 1e-2$) are clearly nice approximations of the optimal one of the original system. Hence, convergence of the underlying shooting method to compute the non-averaged minimizing trajectory is easily obtained. (a) Evolution of n. (b) Evolution of $e_x = e\cos\omega$. (c) Evolution of $e_y = e\sin\omega$.

minimizing trajectory of the real problem sharing the same boundary conditions, thus illustrating the relevance of the continuation method.

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