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Sufficient conditions for time optimality of systems with control on the disk

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Abstract— **The case of time minimization for affine control systems with control on the disk is studied. After recalling the standard sufficient conditions for local optimality in the smooth case, the analysis focusses on the specific type of singularities encountered when the control is prescribed to the disk. Using a suitable stratification, the regularity of the flow is analyzed, which helps to devise verifiable sufficient conditions in terms of left and right limits of Jacobi fields at a switching point. Under the appropriate assumptions, piecewise regularity of the field of extremals is obtained.**

I. INTRODUCTION

In this paper we are interested in the minimum time control of some affine control problems - namely, in dimension four for a rank two distribution. We deal with sufficient conditions for optimality of extremal trajectories of such systems. This topic is a very active field of research, and a variety of different approaches exist and have been applied to a large number of problems. Geometric methods hold an important place in that regard. When the extremal flow is smooth, the theory of conjugate points can be applied, and local optimality holds before the first conjugate time. We recall this result below. The points where the extremal ceases to be globally optimal are cut points, and it is an extremely delicate task to compute cut points and cut loci although some low dimensional situations can be addressed (see, *e.g.*, [5] where an approximation through averaging of the initial problem is studied). Unfortunately, we rarely encounter the smooth case in practice, and there is a lack of general method overcoming the different kind of singularities. An extension of the smooth case method which uses the Poincaré-Cartan integral invariant, see [4], is easier to generalize to non-smooth cases, and has been used to prove local optimality for L^1 minimization of mechanical systems for instance, in [6]. We use a similar technique to prove theorem 3, the main difference being the type of singularity: L^1-minimization of the control creates singularities of codimension one, and the extremal flow is the concatenation of the flows of two regular Hamiltonians. In our case, we have codimension two (and so unstable) singularities, and a Hamiltonian which fails to be Lipschitzian. When the control lies in a box, second order conditions can be of use through a finite dimensional subsystem given by allowing the switching times to variate. Those techniques have been initiated by Stefani and Poggiolini, see for instance [1]. The majority of these works prove local optimality for normal extremal, and a few of them tackle the abnormal case. One can cite for instance [15] where single input systems are handled and can refer as well to [11] where theoretical as well as numerical studies are leaded when the control lies in a polyhedron. We will also tackle only the normal case in the following, since the co-dimension two singularity induced by minimizing the final time is our main focus. The recent paper [3] from Agrachev and Biolo, proves local optimality of these broken extremal around the singularity with extra hypothesis on the adjoint state. Our approach is similar while in a slightly different framework (more suitable for mechanical systems) and easily checked by a simple numerical test. Thanks to this optimality analysis, we can investigate the regularity of a upper bound to the value function of this time optimal problem and prove that it is piecewise smooth.

II. THE SMOOTH THEORY

Let us begin by recalling the classical smooth case. Consider an optimal control system on a manifold M:

$$\begin{cases} \dot{x} = f(x,u),\ u \in U, \\ x(0) = x_0,\ x(t_f) = x_f, \\ C(u) = \int_0^{t_f} \varphi(x(t),u(t))dt \to \min \end{cases} \tag{1}$$

where $U \subset \mathbb{R}^m$, and $f : U \times M \to TM$ is a smooth family of vector field, and $\varphi : M \times U \to \mathbb{R}$ is the cost function. Define $H(x,p,p^0,u) = \langle p, f(x,u)\rangle + p^0\varphi(x,u)$ - $(x,p) \in T^*M$, p^0 a negative number, and u$\in U$ - the pseudo-Hamiltonian associated with (1). By the classical Pontrjagin maximum principle [14], optimal trajectories $x(t)$ associated with an optimal control $u(t)$ are projections of the solutions $(x(t), p(t))$ of the Hamiltonian system associated to H such that, almost everywhere, $H(x(t), p(t), u(t)) = \max_{\tilde{u} \in U} H(x(t), p(t), \tilde{u})$. Solutions of this Hamiltonian system are called extremals. We adapt a method presented in [4], [6] to deal with a codimension one singularity set. Assume now that

$$H^{\max}(x,p) = \max_{u \in U} H(x,p,u)$$

is $\mathscr{C}^2$-smooth and denote $\bar{z}(t) = (\bar{x}(t), \bar{p}(t))$, $t \in [0, \bar{t}_f]$, the extremal starting from $\bar{z}_0 \in T^*M$. Let $\bar{u}$ be the associated control, and consider the variational equation along $\bar{z}(t)$:

$$\dot{\delta z} = J\nabla^2 H(z(t))\delta z \tag{2}$$

where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Solutions of (2) are called Jacobi fields.

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Definition 1 (Conjugate times)**.** A time t_c is called a conjugate time if there exists a Jacobi field δz such that

$$d\pi(z(0))\delta z(0) = d\pi(z(t_c))\delta z(t_c) = 0,$$

that is $\delta x(0) = \delta x(t_c) = 0$, $\pi : T^*M \to M$ being the canonical projection. We say that δz is vertical at 0 and t_c. The point $x(t_c) = \pi(z(t_c))$ is a conjugate point.

The following result implies optimality until the first conjugate time.

Theorem 1. *Assume that*

(i) *The reference extremal $\bar{z}$ is normal,*
(ii) $\frac{\partial x}{\partial p_0}(t,\bar{z}_0) \neq 0$ *for all* $t \in (0,t_f]$.

Then the reference trajectory x is a local minimizer among all the $\mathscr{C}^0$-admissible trajectories with same endpoints.

Assumption (ii) ensures disconjugacy along the reference extremal, and can be verified through a simple numerical test. The proof consists in devising a Lagrangian manifold, and in propagating it using the extremal flow. One can then prove that the projection π is invertible on a suitable submanifold: this allows to lift admissible trajectories with same endpoints to the cotangent bundle, and to compare them using the Poincaré-Cartan invariant. We will extend this proof to the non-smooth case encountered when minimizing time with control on the disk.

III. Setting

Consider the following optimal time control system

$$\begin{cases} \dot{x} = F_0(x) + u_1 F_1(x) + u_2 F_2(x), \\ t \in [0,t_f], \quad |u_1|^2 + |u_2|^2 \leq 1, \\ x(0) = x_0, \quad x(t_f) = x_f, \\ t_f \to \min \end{cases} \tag{3}$$

so that the control set U is the Euclidean disk and the fields F_i are defined on a smooth four dimensional manifold M. We will use the following notation: $[F_i,F_j] := F_{ij}$ for Lie brackets and $\{H_i,H_j\} := H_{ij}$ for Poisson brackets. We denote $\mathscr{U} = L^\infty([0,t_f],U)$ the set of admissible controls, and make the following assumption:

$$\det(F_1(x),F_2(x),F_{01}(x),F_{02}(x)) \neq 0, \quad x \in M. \tag{A1}$$

The (non-smooth) maximized Hamiltonian is

$$H^{\max}(z) = H_0(z) + \sqrt{H_1^2(z) + H_2^2(z)} + p^0,$$

and the singular locus is $\Sigma := \{H_1 = H_2 = 0\}$. One can make a comparison between those singularities and the double switchings obtained by taking $U = [-1,1]^2$ (or even $[-1,1]^m$). It has been proved in [13] that extremals are optimal assuming some strong Legendre-type conditions and the coerciveness of a second variation to a finite dimensional problem obtained by perturbation of the switching times. If this result holds also for the abnormal case, our theorem does not require any coerciveness assumption. The singular set (or *swithching surface*) Σ is partitioned into three subsets as follows:

$$\Sigma_- = \{z \in T^*M \mid H_{12}^2(z) < H_{02}^2(z) + H_{01}^2(z)\},$$
$$\Sigma_+ = \{z \in T^*M \mid H_{12}^2(z) > H_{02}^2(z) + H_{01}^2(z)\},$$
$$\Sigma_0 = \{z \in T^*M \mid H_{12}^2(z) = H_{02}^2(z) + H_{01}^2(z)\}.$$

According to [7] and [12], no regular extremal can reach Σ_+, so all extremals around this set are smooth, and Theorem 1 applies. The singular extremals lying inside cannot be optimal via the Goh condition [9]. According to Pontrjagin's Maximum principle, minimization of the final time implies that normal extremals lie in the level sets $H = 0$, for some $p^0 \leq 0$; normal extremals correspond to $p^0 < 0$ and abnormals to $p^0 = 0$. We will deal with the Σ_- case, which is the most relevant for applications, notably because it contains mechanical systems. We recall the result below from [7]. (See also [2].)

Theorem 2. *In a neighbourhood $O_{\bar{z}}$ with $\bar{z} \in \Sigma_-$, existence and uniqueness of solution for the extremal flow hold, and all extremals are bang-bang, with at most one switch. The extremal flow $z : (t,z_0) \in [0,t_f] \times O_{\bar{z}} \mapsto z(t,z_0) \in M$ is piecewise smooth. More precisely, there is a stratification*

$$O_{\bar{z}} = S_0 \cup S^s \cup S^u \cup \Sigma$$

where

1) *S^s (resp. S^u) is the stable codimension-one submanifold of initial conditions leading to the switching surface (resp. in negative times), $S_0 = O_{\bar{z}} \setminus (S^s \cup \Sigma)$;*
2) *the flow is smooth on $[0,t_f] \times S_0$, and on $[0,t_f] \times S^s \setminus \Delta$ where*
$$\Delta := \{(\bar{t}(z_0),z_0),\ z_0 \in S^s\},$$
and $\bar{t}(z_0)$ is the switching time of the extremal initializing at z_0;
3) *it is continuous on $O_{\bar{z}}$.*

The set S^s is the ensemble of initial conditions brought to the singular locus by the flow, S^u is the set a initial conditions converging to Σ in negative times. In other words, the image of S^s by the flow for times greater than $\bar{t}(z_0)$.

Example. A simple example of such a control system is given by nilpotent approximation of the minimum time Kepler (i.e., two-bodies) problem:

$$\begin{cases} \dot{x}_1 = 1 + x_3 & \dot{x}_3 = u_1 \\ \dot{x}_2 = x_4 & \dot{x}_4 = u_2 \end{cases} \tag{4}$$

with control on the 2-disk, $u_1^2 + u_2^2 \leq 1$.
Let $z_0 \in S^s$.

Proposition 1. *The limits $\dot{z}(\bar{t}(z_0)_\pm, z_0)$ as well as $\frac{\partial z}{\partial z_0}(\bar{t}(z_0), z_0)$ are well defined.*

Proof. Both $\dot{z}(\bar{t}(z_0)_\pm, z_0)$ are easily defined since the control along an extremal has well defined right and left limits at a switching time. Then, thanks to the normal form of Proposition 2 in [7], we know that the map $z_0 \in S^s \mapsto \bar{z}(z_0) := z(\bar{t}(z_0), z_0)$ is smooth. This concludes the proof. □

For extremals outside S^s, the flow of the maximized Hamiltonian is smooth, and the usual sufficient conditions for optimality apply. Let us denote $\bar{z}(t)$ our reference extremal, lying in S^s, with final time $\bar{t}_f$ and $\bar{t} := \bar{t}(z_0)$, $\bar{z}(\bar{t}) := \bar{z}$. We assume that the fiber $T^*_{\bar{x}_0}M$ and S^s intersect transversally:

$$T^*_{\bar{x}_0}M \pitchfork S^s \tag{A2}$$

so $T^*_{\bar{x}_0}M \cap S^s$ is a smooth submanifold of dimension three.

Definition 2 (exponential map)**.** We call exponential mapping from x_0 the application

$$\exp_{\bar{x}_0} : (t, p_0) \in [0, t_f] \times T^*_{\bar{x}_0}M \cap S^s \to \pi(z(t, x_0, p_0)).$$

The exponential map is smooth except on Δ, that is when $x(t, x_0, p_0) \notin \Sigma$. The differential of the exponential mapping $d\exp_{x_0}(t, p_0) = (\dot{x}, \frac{\partial x}{\partial p_0})(t, p_0)$ is a 4×4 matrix, where $\frac{\partial}{\partial p_0}$ denotes the derivation with respect to a set of coordinates on $T^*_{x_0}M \cap S^s$. Set $M(t) := d\exp_{\bar{x}_0}(t, \bar{p}_0)$.

Theorem 3. *Assume that*

(i) *The reference extremal is normal,*

(ii) $\det M(t) \neq 0$ *for all* $t \in (0, \bar{t}) \cup (\bar{t}, \bar{t}_f)$ *and* $\det M(\bar{t}_-) \det M(\bar{t}_+) \neq 0$,

then the reference trajectory is a minimizer among all $\mathscr{C}^0$ neighboring trajectories with same endpoints.

Obviously when $t = 0$, $\frac{\partial x}{\partial p_0}(0, \bar{z}_0) = 0$, and some part of the proof is dedicated to extend condition (ii) to $t = 0$.

IV. Proof of theorem 3

The rest of the paper is devoted to prove Theorem 3.

Lemma 1. *Condition (ii) implies that there exists a Lagrangian submanifold $\mathscr{L}$ transverse to $T^*_{x_0}M$, and close enough to $T^*_{x_0}M$ so $\mathscr{S}_0 = \mathscr{L} \cap S^s$ is a smooth submanifold of dimension 3, and such that $(\frac{\partial x}{\partial t}, \frac{\partial x}{\partial z_0})(t, \bar{z}_0)$ is invertible on $[0, \bar{t}) \times \mathscr{S}_0$, as well as on $(\bar{t}, t_f] \times \mathscr{S}_0$ (z_0 denoting coordinates on $\mathscr{S}_0$).*

Moreover, $(\frac{\partial x}{\partial t}, \frac{\partial x}{\partial z_0})(\bar{t}_-, \bar{z}_0)$ and $(\frac{\partial x}{\partial t}, \frac{\partial x}{\partial z_0})(\bar{t}_+, \bar{z}_0)$ are invertible. Thus, the canonical projection π is a diffeomorphism from $z((0, \bar{t}) \times \mathscr{S}_0)$ onto its image and an homeomorphism from

$$\mathscr{S}_1 = \{z(t, z_0),\ (t, z_0) \in [0, \bar{t}(z_0)] \times \mathscr{S}_0\} \tag{5}$$

onto its image. The same holds true for

$$\mathscr{S}_2 = \{z(t, z_0),\ (t, z_0) \in [\bar{t}(z_0), t_f] \times \mathscr{S}_0\}. \tag{6}$$

Let us prove that π is a homeomorphism on their union. It is sufficient to prove that the extremal crosses transversally $\Sigma_1 := \Sigma \cap \mathscr{S}_1$. Since the map $(t, z_0) \in \mathbb{R} \times \mathscr{S}_0 \mapsto x(t, z_0) \in \pi(\mathscr{S}_1)$ is a homeomorphism, and is differentiable for all $(t, z_0) \neq (\bar{t}(z_0), z_0)$ with well defined limits, we can define its inverse function $z_0(t, x)$, and $f(t, x) = t - \bar{t}(z_0(t, x))$. Thus we have $\Sigma_1 = \{f = 0\}$. Now denote $g(t) = f(t, \bar{x}(t))$, we get

$$\dot{g}(\bar{t}_-) = 1 = \dot{g}(\bar{t}_+) - d\bar{t}(\bar{z}_0)\left[\frac{\partial z_0}{\partial t} + \frac{\partial z_0}{\partial x}\dot{\bar{x}}(t)\right].$$

Since

$$\frac{\partial z_0}{\partial t}(t, z_0(t, x))) = -\left(\frac{\partial x}{\partial z_0}\right)^{-1} \dot{x}(t, z_0(t, x)),$$

we obtain $\dot{g}(\bar{t}_-) = \dot{g}(\bar{t}_+) = 1$. In a neighbourhood of $\bar{z}$, every extremal passes transversely through $\pi(\Sigma_1)$: by restricting $\mathscr{S}_0$ if necessary, every extremal from $\mathscr{S}_0$ passes transversely through $\pi(\Sigma_1)$, and the projection defines a continuous one to one mapping on $\mathscr{S}_1 \cup \mathscr{S}_2$, and even a homeomorphism if we restrict ourselves to a compact neighbourhood of the reference extremal.

We will now prove that the Poincaré-Cartan form $\sigma = pdx - H^{\max}dt$ is exact on $\mathscr{S}_1$ and $\mathscr{S}_2$. Let us first prove that σ is closed on $\mathscr{S}_i$. Tangent vectors to $\mathscr{S}_1$ at z are parametrized as follows:

$$\dot{z}(t, z_0)\delta t + \frac{\partial z}{\partial z_0}(t, z_0)\delta z_0,$$

with $(\delta t, \delta z_0) \in \mathbb{R} \times T_{z_0}\mathscr{S}_0$, $z(t, z_0) = z$, whenever $z \notin \Sigma$. In that last case, tangent vectors are given by

$$\dot{z}(t_-, z_0)\delta t + \frac{\partial z}{\partial z_0}(t_-, z_0)\delta z_0,$$

with $(\delta t, \delta z_0) \in \mathbb{R} \times T_{z_0}\mathscr{S}_0$, $z(t, z_0) = z$. Let $(v_1, v_2) \in T\mathscr{S}_i$, we have

$$\begin{aligned} d\sigma(v_1, v_2) = dp \wedge dx(\frac{\partial z}{\partial z_0}(t, z_0)\delta z_0^1, \frac{\partial z}{\partial z_0}(t, z_0)\delta z_0^2) \\ -dH \wedge dt(v_1, v_2) = \omega(\delta z_0^1, \delta z_0^2) \end{aligned}$$

because the flow is symplectic on S^s, and $dH.\dot{z} = 0$. Eventually, $\omega(\delta z_0^1, \delta z_0^2) = 0$ since $\mathscr{S}_0 \subset \mathscr{L}$ is isotropic. This equality still holds for tangent vectors at $(\bar{t}(z_0), z(\bar{t}(z_0), z_0))$. Being closed, the Poincaré form is actually exact on each $\mathscr{S}_i$. Indeed, consider a curve $\gamma(s) = (t(s), z(t(s), z_0(s)))$ on $\mathscr{S}_1 \cup \mathscr{S}_2$: it retracts continuously on $\gamma_0(s) = (0, z_0(s))$. Then, since σ is closed,

$$\int_\gamma \sigma = \int_{\gamma_0} \sigma,$$

and one can chose $\mathscr{L}$ as the graph of the differential of a smooth function, so

$$\int_{\gamma_0} \sigma = 0$$

by Stokes formula. Let us finally prove that our reference extremal $t \in [0, \bar{t}_f] \mapsto \bar{z}(t) = (\bar{x}(t), \bar{p}(t))$ minimizes the final time among all close $\mathscr{C}^1$-curves with same endpoints. Let $x(t)$, $t \in [0, t_f]$ be a $\mathscr{C}^1$ admissible curve, generated by a control u with $x(0) = x_0$, $\mathscr{C}^0$ close to $\bar{x}$, then, denote $z(t) = (x(t), p(t))$ its well defined lift in $\mathscr{S}_1 \cup \mathscr{S}_2$. Then

$$\int_0^{t_f} p^0 = \int_0^{t_f} p.\dot{x} - H(x, p, u)dt \geq \int_0^{t_f} p\dot{x} - H^{\max}(x, p)dt.$$

Since σ is exact, the right-hand side is actually

$$\int_z \sigma = \int_{\bar{z}} \sigma$$

Thus,

$$t_f p^0 \leq \int_{\bar{z}} \sigma = \bar{t}_f p^0,$$

which proves local optimality for the reference trajectories in the normal case, among all $\mathscr{C}^0$-close curves that have $\mathscr{C}^1$ regularity. A perturbation argument allows us to conclude on optimality with respect to all continuous admissible curves; which ends the proof of Theorem 3.

In the very specific case when $T^*_{x_0}M \subset S^s$, one has to change a bit the exponential mapping defined above, but the same proof basically holds.

Proof of Lemma 1. We follow and adapt the proof in [6]. Let S_0 be a symmetric matrix so that the Lagrangian subspace $L_0 = \{\delta x_0 = S_0 \delta p_0\}$ intersects transversely $T_{\bar{z}_0} S^s$. Consider the two linear symplectic systems

$$\delta \dot{z}(t) = \frac{\partial H^{\max}}{\partial z}(\bar{z}(t))\delta z(t)$$

$t \in [0, \bar{t}[, \delta z(0) = (S_0, I)$ and

$$\dot{\phi}(t) = \frac{\partial H}{\partial z}(\bar{z}(t), u(t))\phi(t),\ t \in [0, \bar{t}[, \phi(0) = I.$$

Set $\delta \tilde{z}(t) = (\delta \tilde{x}(t), \delta \tilde{p}(t)) = \phi(t)^{-1}\delta z(t)$. Since $\delta \tilde{z}(0) = \delta z(0) = (S_0, I)$, the matrix

$$S(t) = \delta \tilde{x}(t) \delta \tilde{p}(t)^{-1}$$

exists for small enough t. It is symmetric since

$$L_t = \exp(X_{H^{\max}} t)'(L_0) \text{ and } (\phi(t))^{-1}(L_t)$$

are Lagrangian submanifolds. One can prove that $\dot{S}(t) \geq 0$ (see [6], appendix), whenever $S(t)$ is defined, as the consequence of the classical first and second order conditions on the maximized Hamiltonian. Then, if $S_0 > 0$ (small enough so that $S(t)$ is defined on $[0, \varepsilon]$), $S(t)$ is invertible, and as such, $\phi(t)^{-1}(L_t) \pitchfork \ker d\pi(\bar{z}_0)$. This implies $L_t \pitchfork \ker d\pi(\bar{z}(t))$ since $\phi(t)(\ker d\pi(\bar{z}_0)) = \ker d\pi(\bar{z}(t))$. There exists a Lagrangian submanifold $\mathscr{L}_0$ of T^*M tangent at L_0 in $\bar{z}_0$. It intersects S^s transversely, and the lemma follows. □

V. Regularity of the field of extremals

Fix $x_0 \in M$, the value function associates to a final state the optimal cost, and is defined as

$$S_{x_0} : x_f \in M \mapsto \inf_{u \in \mathscr{U}} \{t_f,\ x(t_f, u) = x_f\} \in \mathbb{R}.$$

It defines a pseudo-distance between x_0 and x_f and its regularity is a crucial information in optimal control problem, especially in sub-Riemannian geometry where it defines the distance. We give the regularity of the final time for extremals that are locally optimal under the assumptions of the previous section. If they are globally optimal (which holds true for small enough times), this final time coincides with the value function while, otherwise, we only obtain the regularity of an upper bound to the value function. Actually, since the differential equation is homogeneous in the adjoint vector, one can restrict to the unitary bundle of the cotangent bundle ST^*M, and consider

$$\exp : (t_f, p_0) \in \mathbb{R}_+ \times ST^*_{x_0}(M) \mapsto x(t_f, x_0, p_0).$$

The authors have shown in [7] that this function is piecewise smooth, and belongs to the log-exp category. There are two cases:

a) First case: In the neighbourhood of $(x_0, \bar{p}_0) \notin S^s$, the extremal flow, as well as $F(t_f, p_0) := \exp(t_f, p_0) - x_f$, are smooth. If $dF(\bar{t}_f, \bar{p}_0)$ is invertible, that is if $\det(\dot{x}(t, x_0, \bar{p}_0), \frac{\partial x}{\partial p_0}(t, x_0, \bar{p}_0)) \neq 0$, for all t, where p_0 is a system of coordinates on $ST^*(M)$ around $(x_0, \bar{p}_0)$ then, locally, we have a $\mathscr{C}^1$ inverse $F^{-1}(x_f) = (t_f, p_0)(x_f)$. This is the well-known smooth case.

b) Second case: In the neighbourhood $(x_0, \bar{p}_0) \in S^s$, then replace the previous definition of exponential mapping by

$$\exp : (t_f, p_0) \in \mathbb{R}_+ \times S^s \cap T^*_{x_0}M \mapsto x(t_f, x_0, p_0).$$

Under the transversality condition, $S^s \cap T^*_{x_0}M$ is a smooth 3-dimensional submanifold, and since the flow is smooth on S^s, the same process can be applied with the same result, except when $x_f \in \Sigma$. In such a neighbourhood, we only have PC^1 regularity and we need a weaker inverse function theorem. We use a result from [10].

Theorem 4. *Assume*

(i) $\det(\dot{x}(t, \bar{z}_0), \frac{\partial x}{\partial p_0}(\bar{t}_f, \bar{z}_0)) \neq 0$ *for all* $t \neq \bar{t}$

(ii') *the two determinants*

$$\det(\dot{x}(\bar{t}_-, \bar{z}_0), \frac{\partial x}{\partial p_0}(\bar{t}_{f-}, \bar{z}_0)),\ \det(\dot{x}(\bar{t}_+, \bar{z}_0), \frac{\partial x}{\partial p_0}(\bar{t}_{f+}, \bar{z}_0))$$

have the same sign.

Then the final time $x_f \mapsto t_f(x_f)$, is continuous and piecewise $\mathscr{C}^1$ in a neighbourhood of $x(\bar{t}_f, x_0, \bar{p}_0)$.

Proof. Thanks to (i) and (ii') we have a PC^1 inverse, by Theorem 3 in [10], so $x_f \mapsto (t_f(x_f), p_0(x_f))$ is piecewise continuously differentiable. □

Obviously (ii') implies (ii) in Theorem 3, and the extremal is locally optimal. When it is globally optimal, the value function is $S(x_f) = t_f(x_f)$, the final time of the extremal. Otherwise, $S(x_f) \leq t_f$ and we only have PC^1 regularity for an upper bound function to the value function.

VI. Conclusions

We managed to go beyond the smooth case and tackled the case of optimization of the final time, with a Hamiltonian admitting singularities. The approach holds for a large class of control-affine systems that includes mechanical systems. Though the Hamiltonian itself has just Lipschitz regularity, we showed that the tools to prove optimality conditions as well as a disconjugacy hypothesis can still be defined. An interesting development of this work would be the effective computation of switching and conjugate times for problems stemming from mechanical systems.

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