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ORIGINAL ARTICLE

Minimum fuel control of the planar circular restricted three-body problem

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Abstract The circular restricted three-body problem is considered to model the dynamics of an artificial body submitted to the attraction of two planets. Minimization of the fuel consumption of the spacecraft during the transfer, e.g. from the Earth to the Moon, is considered. In the light of the controllability results of Caillau and Daoud (SIAM J Control Optim, 2012), existence for this optimal control problem is discussed under simplifying assumptions. Thanks to Pontryagin maximum principle, the properties of fuel minimizing controls is detailed, revealing a bang-bang structure which is typical of L^1-minimization problems. Because of the resulting non-smoothness of the Hamiltonian two-point boundary value problem, it is difficult to use shooting methods to compute numerical solutions (even with multiple shooting, as many switchings on the control occur when low thrusts are considered). To overcome these difficulties, two homotopies are introduced: One connects the investigated problem to the minimization of the L^2-norm of the control, while the other introduces an interior penalization in the form of a logarithmic barrier. The combination of shooting with these continuation procedures allows to compute fuel optimal transfers for medium or low thrusts in the Earth–Moon system from a geostationary orbit, either towards the L_1 Lagrange point or towards a circular orbit around the Moon. To ensure local optimality of the computed trajectories, second order conditions are evaluated using conjugate point tests.

Keywords Three-body problem · Minimum fuel control · Low-thrust · Indirect methods · Homotopy · Conjugate points

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1 Introduction

The fuel optimal control of space vehicles is a foundational topic in astrodynamics and control. Minimizing propellant usage enables a mission to continue for a longer period of time or, often more crucially, allows for the launch of a less massive spacecraft from the Earth. Thus, even with the advent of highly efficient low-thrust propulsion devices, such as the one used in the recent SMART-1 mission (Racca 2002), the question of how little propellant mass is required for a specific mission remains important. The fundamental theory of optimal control, especially as applied to thrusting space vehicles, has been well established since the 60s in the form of the necessary conditions that an optimal control law will satisfy. This theory is quite remarkable and surprisingly simple to pose. Despite the ease with which it is stated, the solution of specific optimal control problems is quite difficult and continues to be an active area of research. The typical problems that arise in these solutions are mainly rooted in the need to solve two-point boundary value problem for the optimal control to be found. Whereas the numerical solution of initial value problems in ordinary differential equations is well posed and highly advanced, the solution of two-point boundary value problems involving ordinary differential equations is not so well developed. This paper carries out research that directly addresses this aspect of the problem, focusing on transfers in the Earth–Moon system modeled with the circular restricted three-body problem. While practical control laws for transfers in the Earth–Moon system were found using direct methods (Mingotti et al. 2009, 2011; Ross and Scheeres 2007), relying on the knowledge of stable/unstable manifolds of periodic orbits (Martin and Conway 2010), or hybrid ones (Ozimek and Howell 2010)—combination between direct and indirect methods—, the full solution of such problems for the actual thrust constraints using the more rigorous indirect method method is still lacking. The indirect solution of this problem has important theoretical implications as it can be directly checked as to whether it is a local optimum, and in that its solution defines the true mathematical form that extremal solutions to this problem must possess.

The paper is organised as follows: In Sect. 2, the model used for the mathematical analysis of the problem is presented. In Sect. 3, existence issues are sketched; using the necessary condition for solutions, the structure of fuel minimizing controls is derived. Because of the so-called *bang-bang structure* of controls, a continuation technique that deforms the problem into a simpler one is introduced in the last section. Preliminary numerical results using two different continuations with single shooting are finally given.

2 Problem statement

The Earth–Moon system is modeled with the circular restricted three-body problem (Szebehely 1967). An artificial satellite of negligible mass whose thrust is the control is attracted by two primary bodies: The Earth, of mass m_1, and the Moon, of mass m_2. The primaries describe circular orbits around their common center of mass under the influence of their mutual gravitational attraction. The spacecraft motion is supposed to be in the orbital plane containing the primaries (in practice, any deviation manoeuver to get out of this plane calls for large additional energies). The three dimensional case is concerned with a motion of the satellite that does not take place in the plane of motion of the primaries. This case appears when the initial (or final) conditions of the third body (the spacecraft) are such that the body is not initially in the plane of motion of the primaries, or when its initial velocity vector has a component which does not belong to this plane. A standard nondimensionalization of the restricted problem is performed. Since the mass of the third body is negligible, the characteristic mass, $m^* := m_1 + m_2$, is the sum of the two primary masses. The characteristic length is the constant distance between the primaries, l^*. Finally, the characteristic time τ^* is defined in such a way that the gravitational constant G is equal to one. This is accomplished through the use of Kepler's third law:

$$\tau^* = \sqrt{\frac{{l^*}^3}{Gm^*}}.$$

Let $\mu \in (0, 1)$ be the ratio of the primary masses, $\mu := m_2/m^*$, and let us denote m the mass of the spacecraft and $T_{\max}$ its maximal thrust. The position-speed vector x is in $\mathbf{R}^4$. The spacecraft motion is ruled by the following first-order controlled differential equation (Caillau et al. 2010, 2011):

$$\begin{aligned}\dot{x}(t) &= f(x(t), u(t))\\ &= F_0(x(t)) + \frac{T_{\max}}{m} F_1(x(t))u_1(t) + \frac{T_{\max}}{m} F_2(x(t))u_2(t),\\ |u(t)| &= \sqrt{u_1^2(t) + u_2^2(t)} \leq 1\end{aligned}$$

where

$$F_0(x) = \begin{bmatrix} x_3 \\ x_4 \\ 2x_4 + x_1 - \frac{1-\mu}{r_{13}^3}(x_1 + \mu) - \frac{\mu}{r_{23}^3}(x_1 - 1 + \mu) \\ -2x_3 + x_2 - \frac{1-\mu}{r_{13}^3} x_2 - \frac{\mu}{r_{23}^3} x_2 \end{bmatrix},$$

$$F_1(x) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad F_2(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

and $u = (u_1, u_2)$ the control vector. The spacecraft dynamics are written in barycentric rotating frame with nondimensional units: The angular velocity of the primaries, their distance, and the sum of their masses are all set to 1. The first primary of mass $1 - \mu$, is located at $(-\mu, 0)$, whereas the second primary, of mass μ, is located at $(1 - \mu, 0)$. The quantities r_{13} and r_{23} are the distances between the spacecraft and, the Earth and Moon respectively: $r_{13} = ((x_1 + \mu)^2 + x_2^2)^{1/2}$ and $r_{23} = ((x_1 - 1 + \mu)^2 + x_2^2)^{1/2}$. Excluding the singularities at the two primaries, one defines the state space as the submanifold

$$X_\mu := \{x \in \mathbf{R}^4 \mid (x_1, x_2) \neq (-\mu, 0) \quad \text{and} \quad (x_1, x_2) \neq (1 - \mu, 0)\}.$$

Remark 1 One has to multiply the ratio $T_{\max}/m$—where $T_{\max}$ is expressed in Newtons and m in kilograms—by the normalization constant ${l^*}^3/(Gm^*)$ when performing numerical computations.

The drift F_0 describes the dynamics of the uncontrolled motion. It has five equilibrium points, namely the Lagrange points or *libration points*, $L_1, \ldots, L_5$. Their locations are calculated by solving $F_0(x) = 0$. There are three collinear points situated on the the primaries axis and two points forming symmetric equilateral triangles with the two primaries. In this study, we are interested in the L_1 collinear libration point situated between the primaries.[1] The dynamics should also take into account the variation of the mass according to

$$\dot{m}(t) = -\beta|u(t)|$$

where β depends on the specific impulse of the engine. As a result, optimizing fuel comsuption amounts to minimizing the L^1-norm of the control,

$$\int_0^{t_f} |u(t)|\, \mathrm{d}t \to \min,$$

where t_f is the fixed final time and $|.|$ the Euclidian norm in $\mathbf{R}^2$. The variation of the mass will be neglected in our model for the following reasons: (i) On typical examples, the mass varies only slightly during the transfer (this fact is verified numerically); (ii) In the two body problem the numerical results are qualitatively unchanged whether the dynamics of the mass is taken into account or not for minimum time transfers (Caillau et al. 2003; Bonnard et al. 2007) or minimum fuel transfers (Gergaud and Haberkorn 2006) and typical specific impulses; (iii) Mathematically, the inclusion of the mass dynamics leads to a more complicated system of equations, therefore the system is reduced to equations of motion alone. (iv) Once a solution of the fixed mass problem is calculated, a continuation procedure (see Sect. 4) on the parameter β may used to to connect the simplified problem to the one with mass variation.

For a fixed final time t_f, the control u achieving the minimum fuel transfer is the solution of the optimal control problem

$$(P) \begin{cases} J(u) := \int_0^{t_f} |u(t)|\, \mathrm{d}t \to \min \\ \dot{x}(t) = f(x(t), u(t)), \quad |u(t)| \leq 1 \\ x(0) \in X_0, \quad x(t_f) \in X_f \end{cases}$$

where X_0 and X_f are submanifolds of $X_\mu \subset \mathbf{R}^4$. In this paper, we investigate transfers from an Earth orbit. Two targets are considered: First, the libration point L_1; secondly, an orbit around the Moon that we call MO.

3 Existence and structure of controls

Controllability properties are studied in Caillau and Daoud (2012) where they are related to the value of the Jacobi first integral J_μ at the L_1 Lagrange point. Recall that

$$\left[J_\mu(x) = \frac{1}{2}(x_3^2 + x_4^2) - \frac{1}{2}(x_1^2 + x_2^2) - \frac{1-\mu}{\sqrt{(x_1+\mu)^2 + x_2^2}} - \frac{\mu}{\sqrt{(x_1 - 1 + \mu)^2 + x_2^2}} \right],$$

and assume that points in X_0 and X_f have Jacobi constant less than[2] $J_\mu(L_1)$; then, for any positive $T_{\max}$, an admissible trajectory connecting X_0 and X_f exists.[3] Despite the existence of admissible trajectories, existence of minimizing controls is still an issue essentially because

[1] We do not follow here the classical notation of, e.g., Szebehely (1967), where this point is referred to as the L_2 libration point.

[2] Note that this value depends on μ not only through J_μ but also through L_1.

[3] To be accurate, one has to restrict to the appropriate connex component of $\{x \in X_\mu \mid J_\mu(x) < J_\mu(L_1)\}$ (see Caillau and Daoud 2012).

of collisions, as one has to prove that trajectories not remaining in a fix compact cannot be optimal. We do not address this delicate point here and assume that we can restrict to some compact (depending on $\mu,\ T_{\max},\ X_0$ and X_f) to which optimal trajectories are interior.

Proposition 1 *Under the previous assumption, existence of optimal trajectories hold.*

Proof As we have assumed that trajectories remain in a fix compact, existence is given by Filippov theorem (Agrachev and Sachkov 2004) since the convexity issues due to the $|u|$ term in the integrand of the cost can be dealt with as in Gergaud and Haberkorn (2006). □

Minimizing trajectories are projections of extremal curves parameterized by the maximum principle (Agrachev and Sachkov 2004). Let $\bar{u} : [0, t_f] \to \mathbf{R}^2$ be a measurable optimal control, and let $\bar{x}$ be the associated trajectory; there exist a nonpositive scalar $\bar{p}^0$ and a Lipschitz covector function $\bar{p} : [0, t_f] \to (\mathbf{R}^n)^*$, not both zero, such that

$$\dot{\bar{x}}(t) = \frac{\partial H}{\partial p}(\bar{x}(t), \bar{u}(t), \bar{p}(t)), \quad \dot{\bar{p}}(t) = -\frac{\partial H}{\partial x}(\bar{x}(t), \bar{u}(t), \bar{p}(t))$$

almost everywhere on $[0, t_f]$ where $H(x, u, p) := p^0 f^0(x, u) + pf(x, u)$. Moreover, the maximization condition holds,

$$H(\bar{x}(t), \bar{u}(t), \bar{p}(t)) = \max_{|u| \le 1} H(\bar{x}(t), u, \bar{p}(t))$$

almost everywhere on $[0, t_f]$.

The triple $(\bar{x}, \bar{u}, \bar{p})$ is called an extremal. Finally, transversality conditions assert that

$$\bar{p}(0) \perp T_{x(0)} X_0, \quad \bar{p}(t_f) \perp T_{x(t_f)} X_f. \tag{1}$$

The nontrivial pair $(\bar{p}^0, \bar{p})$ is defined up to a positive scalar, and there are two cases: Normal ($\bar{p}^0 < 0$) and abnormal ($\bar{p}^0 = 0$). For given $\mu,\ T_{\max},\ X_0$ and X_f, let $t_f^{\min}$ denote the minimum time (which exists under the same assumption as in Proposition 1).

Proposition 2 *If $t_f > t_f^{\min}$ there is no abnormal trajectory.*

Proof Let $\bar{u}$ be an optimal control. Along the extremal associated with $\bar{u}$, define

$$\varphi(t) := (H_1, H_2)(\bar{x}(t), \bar{p}(t)), \quad H_i(x, p) := \langle p, F_i(x, p)\rangle, \quad i = 0, 2. \tag{2}$$

This function has a finite number of zeros on $[0, t_f]$ (see, e.g., Caillau et al. 2003). Assume by contradiction that $\bar{p}^0 = 0$; the maximization condition implies that $\bar{u}(t) = \varphi(t)/|\varphi(t)|$ almost everywhere, so $|\bar{u}(t)| = 1$ almost everywhere. As a consequence,

$$\int_0^{t_f} |\bar{u}(t)|\,\mathrm{d}t = t_f > t_f^{\min}$$

which contradicts optimality as $\widetilde{u}$ defined as equal to the minimum time control on $[0, t_f^{\min}]$ and 0 on $[t_f^{\min}, t_f]$ has a smaller performance index. □

Hence, we are only interested in the normal case $\bar{p}^0 \neq 0$, and we normalize $(\bar{p}^0, \bar{p})$ by setting $\bar{p}^0 = -1$. The optimal control is calculated thanks to the maximization condition. We introduce the switching function

$$\psi(x, p) := \frac{T_{\max}}{m}|\varphi(x, p)| - 1. \tag{3}$$

The control value is determined according to the sign of ψ as below:

$$u(x,p) = \begin{cases} \varphi(x,p)/|\varphi(x,p)| & \text{if } \psi(x,p) > 0, \\ 0 \quad \text{if } \psi(x,p) < 0, \\ \alpha\varphi(x,p)/|\varphi(x,p)|,\ \alpha \in [0,1] & \text{if } \psi(x,p) = 0. \end{cases}$$

The maximized Hamiltonian is a function of (x, p) only, $H(x, p) = H_0(x, p) + \psi_+(x, p)$ with $y_+ := \max(y, 0)$. The maximum principle thus leads to solve the two point boundary value problem whose right-hand side is well defined outside $\{\varphi = 0\} \cup \{\psi = 0\}$,

$$\dot{x}(t) = \frac{\partial H}{\partial p}(x(t), p(t)), \quad \dot{p}(t) = -\frac{\partial H}{\partial x}(x(t), p(t)), \tag{4}$$
$$b_0(x(0), p(0)) = 0, \quad b_f(x(t_f), p(t_f)) = 0,$$

where the functions b_0 and b_f parameterize the transversality conditions (1). The resolution of this boundary value problem amounts to finding a zero of the shooting function $S : \mathbf{R}^8 \to \mathbf{R}^8$ defined by:

$$S(x_0, p_0) := (b(x_0, p_0), b(x(t_f, x_0, p_0), p(t_f, x_0, p_0)))$$

where $(x, p)(t, x_0, p_0)$ denotes the solution of the Hamiltonian differential equation (4) at time t. The true Hamiltonian is non smooth and only continuous. The optimal control is bang-bang: $|u|$ switches between zero and one. As in the two-body case, this fact makes it difficult if not impossible, to find zeros of the shooting function which is not even continuous at some points (Gergaud and Haberkorn 2006). This is due to the fact that the cost function, $f^0 : (x, u) \mapsto |u|$, is only concave but not strictly concave with respect to u, and so is the Hamiltonian $H(x, u, p)$ with respect to u. The initialization task is complicated since the control solution involves switchings. In addition, the number of switchings increases when considering lower thrusts. To have an initial guess to solve the shooting equation with a Newton-type method requires to know *a priori* the number and approximate location of the optimal control switchings. To address this concern, we use continuation methods as in the two-body case (Gergaud and Haberkorn 2006).

4 Continuation methods

We embed the previous optimal control into a one-parameter family of control problems (P_λ) with strictly concave cost integrand $f_0(x, u, \lambda)$,

$$(P_\lambda) \begin{cases} J_\lambda(u) := \int_0^{t_f} f^0(x(t), u(t), \lambda)\, \mathrm{d}t \to \min \\ \dot{x}(t) = f(x(t), u(t)), \quad |u(t)| \le 1 \\ x(0) \in X_0, \quad x(t_f) \in X_f \end{cases}$$

The resolution of (P_λ), for the two transfer cases, leads to find a zero of a shooting function, S_λ. The homotopy on the parameter $\lambda \in [\lambda_0, \lambda_f]$ is choosen such that:

- (i) For $\lambda = \lambda_f$ we retrieve the initial problem, $(P_{\lambda_f}) = (P)$.
- (ii) It is known how to find a zero of S_{λ_0}, which solves (P_{λ_0}).
- (iii) The cost function is strictly concave with respect to u for $\lambda < \lambda_f$ and so is the maximized Hamiltonian; as a result, the function $u(x, p)$ is at least continuous.

Once a zero of S_{λ_0} is found, the path of zeros of the shooting function S_λ is followed from $\lambda = \lambda_0$ to λ_f.

4.1 Energy-consumption homotopy

We focus first on a so-called energy-consumption homotopy, also referred to as $L^2 - L^1$ homotopy as it connects the minimization of the (squarred) L^2-norm of the control to the L^1-norm. The homotopy is obtained by taking a convex combination of the two costs,

$$f^0(x, u, \lambda) := (1-\lambda)|u|^2 + \lambda|u|.$$

When $\lambda = 1$, problem (P) is retrieved. When $\lambda = 0$ the problem is called *energy minimization* problem. It is known to be much easier to solve than the minimum fuel problem, as investigated in the two-body case (Gergaud and Haberkorn 2006). The Hamiltonian now is

$$H_\lambda(x, u, p) = -\lambda|u| - (1-\lambda)|u|^2 + H_0(x, p) + \frac{T_{\max}}{m}(u_1 H_1(x, p) + u_2 H_2(x, p)).$$

For $\lambda < 1$, $H_\lambda(x, u, p)$ is strictly concave with respect to u and admits a unique maximizer continuously depending on (x, p). Let us define

$$\alpha_\lambda(x, p) := \frac{(T_{\max}/m)|\varphi(x, p)| - \lambda}{2(1-\lambda)}, \quad \lambda < 1.$$

As before $|u| \leq 1$, so the maximization condition implies that

$$u_\lambda(x, p) = \begin{cases} \varphi(x, p)/|\varphi(x, p)| & \text{if } \alpha_\lambda(x, p) > 1, \\ 0 & \text{if } \alpha_\lambda(x, p) < 0, \\ \alpha_\lambda(x, p)\varphi(x, p)/|\varphi(x, p)| & \text{if } 0 \leq \alpha_\lambda(x, p) \leq 1. \end{cases}$$

Remark 2 The optimal control u_λ is continuous but not smooth so the true Hamiltonian in this case is also only continuous and non smooth.

In order to study the variation of the criterion $J_\lambda(u)$ with respect to the homotopic parameter λ and its convergence when λ tends towards 1, we recall the following result.

Proposition 3 (Gergaud and Haberkorn 2006) *Let (x_λ, u_λ) be a solution of the problem (P_λ) then for $0 \leq \lambda \leq \lambda' \leq 1$, we have:*

(i) $J_\lambda(u_\lambda) \leq J_{\lambda'}(u_{\lambda'}) \leq J_1(u_1) \leq J_1(u_\lambda)$
(ii) $J_\lambda(u_\lambda)$ *and* $J_1(u_\lambda)$ *tend to* $J_1(u_1)$ *when* λ *tends to* 1.

To start the energy-consumption homotopy one has to solve the minimum energy transfer problem. To facilitate the numerical resolution, we omit the constraint on the control, $|u| \leq 1$. This makes the control and the true Hamiltonian smooth. Let us consider so

$$(\widetilde{P}_0) \begin{cases} \int_0^{t_f} |u(t)|^2 \, \mathrm{d}t \to \min \\ \dot{x}(t) = f(x(t), u(t)) \\ x(0) \in X_0, \quad x(t_f) \in X_f \end{cases}$$

One has $(\widetilde{P}_0) = (P_0)$ when the control solution verifies $|u(t)| \leq 1$, $\forall t \in [0, t_f]$. Applying the maximum principle to the relaxed problem leads to a smooth maximized Hamiltonian:

$$\widetilde{H}_0(x, p) = H_0(x, p) + \frac{T_{\max}^2}{2m^2}(H_1^2(x, p) + H_2^2(x, p)).$$

The strategy is (i) to solve $\widetilde{P}_0$—which is quite easy even without *a priori* knowledge on the solution—, (ii) then to increase the fixed final time (performing a discrete continuation on t_f)

Table 1 Minimum energy transfer towards the orbit MO, $T_{\max} = 10\,\mathrm{N}$

c_{t_f}	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7
$\|u\|_\infty$	4.12	2.02	1.22	1.02	1.15	1.49	0.83	0.72

The ratio $c_{t_f} := t_f/t_f^{\min}$ between the fixed final time and the minimum time of the CR3BP computed in Caillau and Daoud (2012) is iteratively increased (discrete continuation on c_{t_f}) until the constraint on the control is satisfied

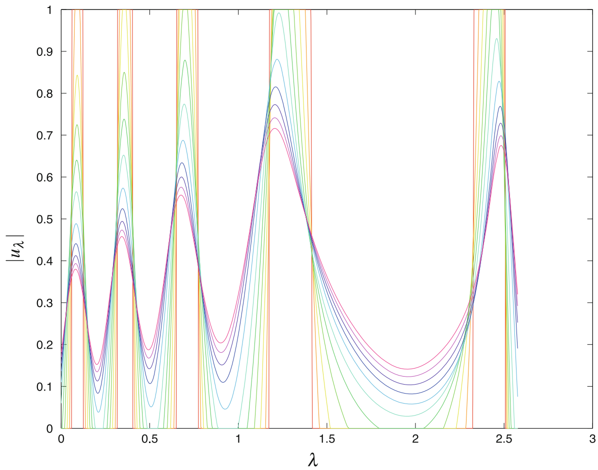


Fig. 1 Control norm $|u_\lambda|$ versus time for λ equal to 0.0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, and 0.99 (*red*), L^2–L^1 homotopy, target MO, $T_{\max} = 10\,\mathrm{N}$. Convergence is observed numerically when λ tends to one

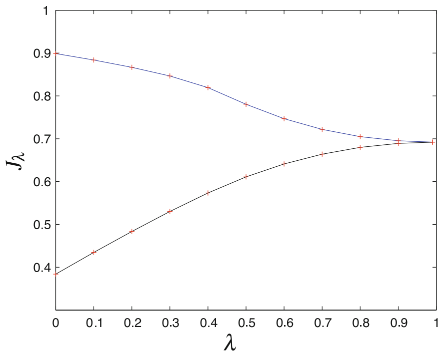


Fig. 2 Value function versus time for λ equal to 0.0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9 and 0.99 (*red*), L^2–L^1 homotopy, target MO, $T_{\max} = 10\,\mathrm{N}$. Convergence is observed numerically when λ tends to one

so as to satisfy the constraint on the control (the essential supremum of the control tends to zero as $t_f \to \infty$, see Bonnard et al. 2010). The approach detailed previously is consolidated by numerical tests. Let us consider a spacecraft of mass $m = 1500\,\mathrm{kg}$. We aim to design a minimum fuel transfer from a circular orbit around the Earth of radius 42.165 Mm towards a circular orbit MO around the Moon of radius 13.084 Mm. The maximum thrust $T_{\max}$ is 10 N. Table 1 summarizes the numerical results obtained for $(\widetilde{P}_0)$. The final time for the L^2–L^1 continuation is so set to $1.7 \times t_f^{\min}$. The bang-bang structure on $|u|$ is captured by the method as λ tends to 1 (see Figs. 1, 2). Additional results as well as corresponding trajectories in the moving frame for thrusts between 10 and 3 N are given Fig. 3.

The design of space missions requires a compromise between the mission duration to reach the target and the propellant consumption. In this context, a very important informa-

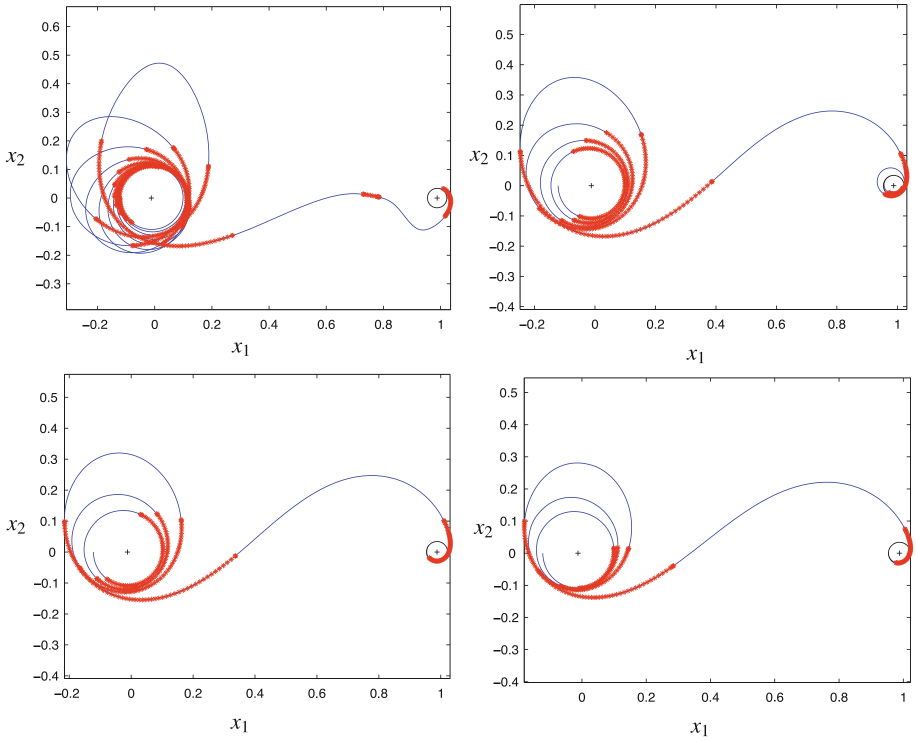
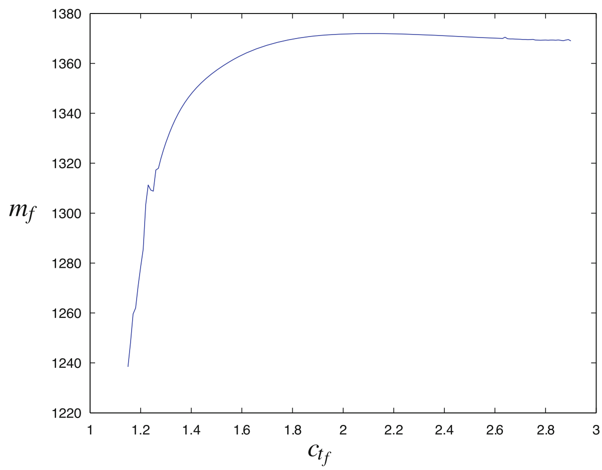


Fig. 3 Minimum fuel consumption optimal trajectory to MO, energy-consumption homotopy, moving frame. *Top left* $T_{\max} = 10\,\text{N}$, *top right* $T_{\max} = 7\,\text{N}$, *bottom left* $T_{\max} = 5\,\text{N}$, and $T_{\max} = 3\,\text{N}$ *bottom right*. *Red* points indicate thrust arcs. The number of these arcs increases when the maximal thrust decreases, reflecting the growing number of revolutions around the primaries

Fig. 4 Maximum final mass (in kilograms) with respect to the time ratio $c_{t_f} = t_f / t_f^{\min}$. Transfer towards MO, maximum thrust of 10 N. The final mass is comprised between 1360 and 1370 kg when $t_f \geq 1.5 t_f^{\min}$; the final mass is almost constant for $c_{t_f} \geq 2$



tion is the graph giving the variation of the final mass versus the transfer time. Such a graph, as presented Fig. 4, is obtained using a continuation on the ratio c_{t_f}; the graph is concave, indicating that a good compromise is obtained without going to large values of c_{t_f}.

The L^2-L^1 continuation has two drawbacks: Firstly, the precision on the solution of the shooting function deteriorates significantly when λ tends to one as the number of control switchings becomes very large for low thrusts. Secondly, we are not able to test local optimality of the computed extremals because second order conditions are only valid for the smooth case (here, the maximized Hamiltonian is only continuous). A more regular homotopy is introduced in the next subsection to try and overcome these difficulties.

4.2 Logarithmic barrier homotopy

The logarithmic barrier homotopy is defined by the following new one parameter family of problems ($\varepsilon > 0$):

$$(P_\varepsilon)\begin{cases} J_\varepsilon(u) := \int_0^{t_f} |u(t)| - \varepsilon \log|u(t)| - \varepsilon \log(1-|u(t)|)\,\mathrm{d}t \to \min \\ \dot{x}(t) = f(x(t), u(t)), \quad 0 < |u(t)| < 1 \\ x(0) \in X_0, \quad x(t_f) \in X_f \end{cases}$$

whereas the original cost functional is retrieved when $\varepsilon = 0$, the control domain is the pointed open unit ball $\{u \in \mathbf{R}^2 \mid 0 < u_1^2 + u_2^2 < 1\}$ for any ε. This domain not being convex, existence is an issue we do not elaborate on here. When applying the maximum principle, one considers the Hamiltonian

$$H_\varepsilon(x, u, p) = -|u| + \varepsilon(\log(|u|) + \log(1-|u|)) + H_0(x, p) + \frac{T_{\max}}{m}(u_1 H_1(x, p) + u_2 H_2).$$

For any positive ε, the unique maximizer is

$$u_\varepsilon(x, p) = \alpha_\varepsilon(x, p)\frac{\varphi(x, p)}{|\varphi(x, p)|}$$

with ψ and φ defined by (2) and (3), respectively, and

$$\alpha_\varepsilon(x, p) := \frac{2\varepsilon}{2\varepsilon - \psi(x, p) + \sqrt{\psi(x, p)^2 + 4\varepsilon^2}}.$$

The maximized Hamiltonian reads

$$H_\varepsilon(x, p) = H_0(x, p) + \psi(x, p)\alpha_\varepsilon(x, p) + \varepsilon \log \alpha_\varepsilon(x, p) + \varepsilon \log(1 - \alpha_\varepsilon(x, p)).$$

Although nonconvexity makes the convergence analysis more intricate as in the L^2-L^1 case, we recall some monotonicity properties of the value function.

Proposition 4 (Gergaud 2008) *For all $0 < \varepsilon \leq \varepsilon'$, if u, u_ε and $u_{\varepsilon'}$ are respectively the optimal controls solution of the problems (P), (P_ε) and $(P_{\varepsilon'})$, then*

$$J(u) \leq J(u_\varepsilon) \leq J_\varepsilon(u_\varepsilon) \leq J_{\varepsilon'}(u_{\varepsilon'}).$$

Following an idea of Bertrand and Epenoy (2002), the numerical strategy is to find ε_0 such that the solution of $(\widetilde{P}_0)$ (relaxed L^2-minimization problem) is a suitable guess for (P_{ε_0}). Then, since the maximized Hamiltonian is now smooth, we can use differential continuation as implemented in the `hampath` code (Caillau et al. 2012) to solve the shooting equation for ε close enough to zero. The result of this approach to compute a minimum fuel trajectory from the geostationary orbit towards the L_1 Lagrange point is given Fig. 5. The maximum thrust is $T_{\max} = 0.3\,\mathrm{N}$ for a mass $m = 1500\,\mathrm{kg}$. The transfer to the L_1 point is performed in 8 months and 5 days with a fuel consumption of 31 kg. In terms of acceleration, this case is equivalent to the SMART-1 mission (see http://www.esa.int/export/SPECIALS/

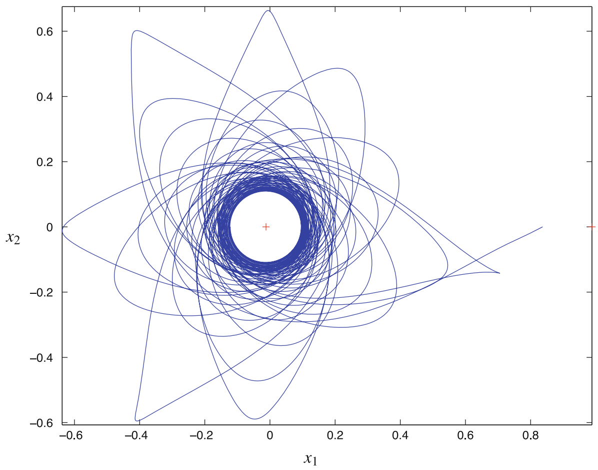


Fig. 5 Minimum fuel consumption trajectory from a geostationary orbit towards L_1, moving frame, logarithmic barrier homotopy, L_1 target, $T_{\max} = 0.3\,\text{N}$

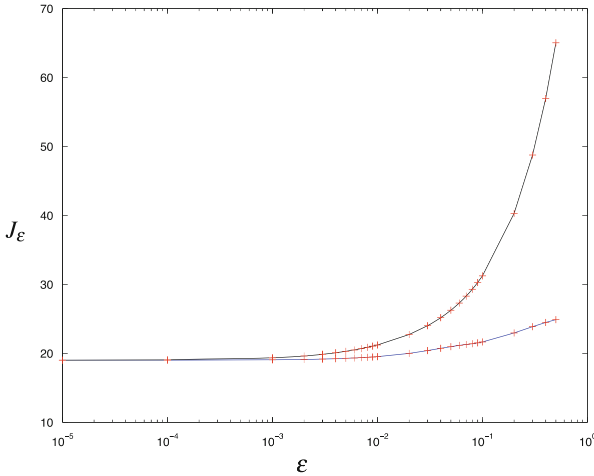


Fig. 6 Value functions $J(u_\varepsilon)$ (*blue*) and $J_\varepsilon(u_\varepsilon)$ (*black*), logarithmic barrier homotopy, L_1 target, $T_{\max} = 0.3\,\text{N}$. The red points indicate the values actually computed. Both $J(u_\varepsilon)$ and $J_\varepsilon(u_\varepsilon)$ decrease to the same limit value, which is consistent with Proposition 4

SMART-1) for which the maximum thrust was 0.07 N for a mass of 350 kg. In the SMART-1 case, initial and final orbits are different from the ones in our simplified model, and the transfer is not planar so only a qualitative and preliminary comparison can be drawn. The duration of the SMART-1 mission was 13 months for a fuel consumption of 59 kg. For low thrust cases the transfer time is high and the number of oscillations of the norm of the optimal control increases a lot. Small homotopic steps are thus needed to ensure shooting convergence down to low ε values (see Fig. 6).

As afore-mentioned, the maximum principle is essentially a first order necessary condition. To ensure local optimality, second order necessary conditions are introduced. We recall the theoretical framework and explain the suitable numerical tests. Let $(\bar{x}, \bar{u}, \bar{p})$ be a reference extremal such that $\bar{z} = (\bar{x}, \bar{p})$ is solution of the smooth maximized Hamiltonian system

$$\dot{z}(t) = \overrightarrow{H}_\varepsilon(z(t)), \quad t \in [0, t_f]$$

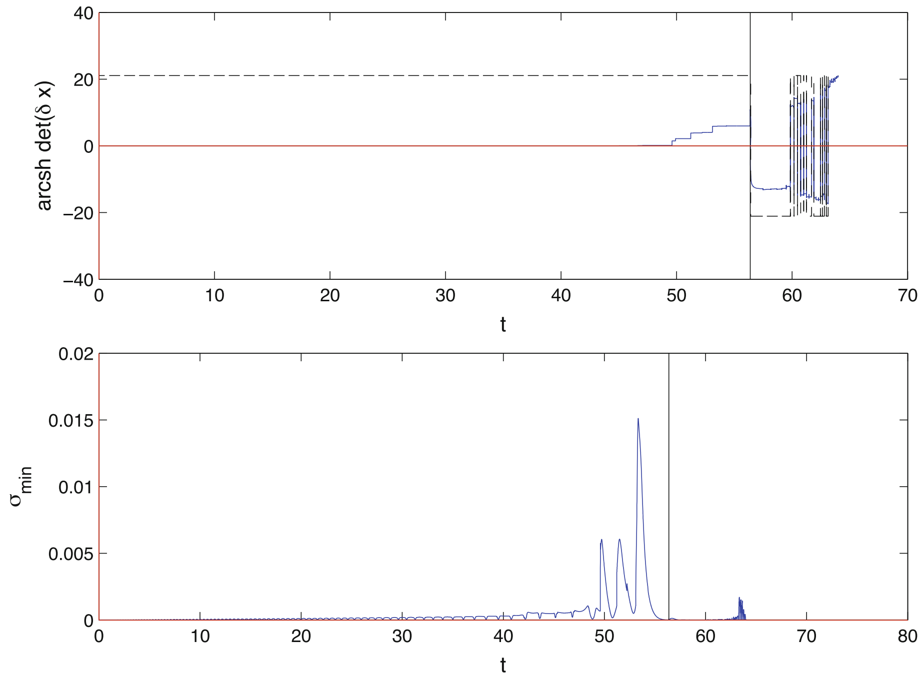


Fig. 7 Second order condition test for a minimum energy transfer towards the L_1 Lagrange point, $T_{\max} =$ 0.3 N (log-barrier homotopy, $\varepsilon = 10^{-5}$). On the *top* graph is displayed the determinant of the four Jacobi fields required for the rank test (log-like scale on the y-axis); the first zero of the determinant (*dashed vertical line*)—that is the first conjugate time—is located slightly after the final time (*plain vertical line*). A complementary test consists in computing the singular value decomposition of the matrix formed by $\delta x_i(t),\ i = 1, \ldots, 4$; see *bottom* graph

where $\overrightarrow{H}_\varepsilon = (\partial H_\varepsilon/\partial p, -\partial H_\varepsilon/\partial x)$, and satisfies the initial and final conditions $x(0) = x_0$ and $x(t_f) = x_f$ (we first assume boundary conditions given as endpoints). The variational or *Jacobi* equation along the extremal is the linearized system

$$\delta\dot{z}(t) = \overrightarrow{H}'_\varepsilon(\bar{z}(t))\delta z(t), \quad t \in [0, t_f].$$

A *Jacobi field* is a non trivial solution $\delta z = (\delta x, \delta p)$ of this equation; it is said to be *vertical* at time t if $\delta x(t) = \Pi \delta z(t)$ (where $\Pi(x, p) = x$ is the canonical projection for $\mathbf{R}^8$ to $\mathbf{R}^4$) vanishes. A positive time t_c is *conjugate* (to $t = 0$) if there exists a Jacobi field along the reference extremal which is vertical at $t = 0$ and $t = t_c$. The point $x(t_c)$ is then called a *conjugate point*. This notion generalizes the notion of conjugacy of Riemannian geometry and is related to local optimality of trajectories. Before stating the generalization of the Jaocbi theory to the optimal control setting, recall that an extremal is said to be *regular* if the strong Legendre condition holds,

$$\frac{\partial^2 H_\varepsilon}{\partial u^2}(\bar{x}(t), \bar{u}(t), \bar{p}(t))(u, u) \le -\alpha |u|^2, \quad u \in \mathbf{R}^2, \quad t \in [0, t_f],$$

for some positive constant α.

Lemma 1 *Extremal solutions of* (P_ε) *are regular.*

Proof Setting $\rho = |u|$, it is sufficient to prove that the Hessian of H_ε with respect to ρ is negative definite. Now,

$$\frac{\partial^2 H_\varepsilon}{\partial \rho^2} = -\varepsilon \left(\frac{1}{\rho^2} + \frac{1}{(1-\rho)^2} \right) < 0$$

along the reference extremal, so the result follows. □

Proposition 5 *If there is no conjugate point on* $(0, t_f]$ *along the reference extremal, then the extremal is* $\mathscr{C}^0$*-locally optimal.*

Proof The maximized normal Hamiltonian is well defined and smooth, and the reference extremal is regular by virtue of Lemma 1; in the absence of conjugate point on $(0, t_f]$, we can construct a field of extremals as in Agrachev and Sachkov (2004) and conclude on optimality of the reference trajectory among $\mathscr{C}^0$-close trajectories with same endpoints. □

The algorithm to evaluate conjugate times is the following Bonnard et al. (2005): One has to compute numerically the Jacobi fields $\delta z_i = (\delta x_i, \delta p_i),\ i = 1, \ldots, 4$, corresponding to the initial conditions $\delta x_i(0) = 0$ and $\delta p_i(0) = e_i,\ i = 1, \ldots, 4$, where $(e_i)_{i=1,\ldots,4}$ is the canonical basis of $\mathbf{R}^4$. The time t_c is conjugate time whenever

$$\text{rank}\{\delta x_1(t_c), \delta x_2(t_c), \delta x_3(t_c), \delta x_4(t_c)\} < 4.$$

Such a test is implemented in the numerical code `hampath` (Caillau et al. 2012). The numerical rank tests show that the first conjugate time, t_{1c}, along the computed extremals occurs after the final time, ensuring local optimality (see Fig. 7 for $T_{\max} = 0.3\,\text{N}$).

5 Conclusion

A method to design minimum fuel, low thrust, transfers in the Earth–Moon system has been described in this paper. The method uses Pontryagin maximum principle to compute the optimal law realizing the transfer in the circular restricted three-body problem. Convergence issues of the shooting algorithm are overcomed by means of suitable continuations, and two such homotopies have been proposed. Moreover, second-order conditions are tested to ensure local optimality of the computed extremals. Although these results on a simplified two-dimensional model are of preliminary nature, they illustrate how a combination of shooting and homotopic techniques allow to address efficiently three-body control problems. Using again continuation from the simplified model should permit to compute realistic transfers in more accurate and higher dimensional models (see for instance the 3D results for minimum time in Caillau and Daoud 2012; Daoud 2011).

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