ON THE CONTROLLABILITY OF NONLINEAR SYSTEMS WITH A PERIODIC DRIFT

J.-B. CAILLAU, L. DELL'ELCE, A. HERASIMENKA, J.-B. POMET

Abstract. Sufficient conditions are established for controllability of affine control systems with a drift all of whose solutions are periodic. In contrast with previously known results, these conditions encompass the case of a control set whose convex hull is not a neighbourhood of the origin. The condition is expressed by means of pushforwards along the flow of the drift, rather than in terms of Lie brackets. It turns out that this also amounts to local controllability of a time-varying linear approximation with constrained controls. Global and local results are given, as well as a few illustrative examples.

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Introduction

This paper is devoted to controllability properties of affine control systems, defined by a drift vector field and $m \geq 1$ control vector fields, in the case where all solutions of the drift vector field (behavior with a zero control) are periodic, and the control is constrained to a convex set that contains zero, but is not necessarily a neiborhood of zero. A particular case of such systems are solar sails, which are at the core of motivation of this study, see references [7, 8][1] by the authors. Governed by solar radiation pressure, solar sails are only capable of generating forces contained in a convex cone of revolution around the direction of the incoming light; forces being the controls and zero being the vertex of this cone, this is a typical case where zero is in the boundary of the control set.

Devising conditions on the vector fields (and the set U) for controllability is an old problem. Most known controllability sufficient conditions assume that the vector fields are bracket generating (full rank of the distribution spanned by the Lie algebra they generate), which is indeed necessary, and prove controllability under an additional assumption on the drift vector field; this occurs, for instance if the drift vector field is zero [13], or if all orbits of the drift vector field are periodic, or if the drift vector field is Poisson stable [2]. See also textbooks like [9] or [1]. These results however require that zero belongs to the interior of the convex hull of U. Motivated by the study of the controllability of non-ideal solar sails, as explained above, the present paper investigates controllability in the case where zero is in U, but is not contained in the interior of its convex hull (the results are still valid if U happens to be a neighborhood of zero, but would be obtained in a simpler way from known results in that case). It is based on sufficient conditions that are not expressed in terms of Lie brackets of vector fields but on some construction attached both to the vector fields and to the set U. The running assumption on the drift vector field, when dealing with global controllability, is that all its orbits are periodic, which implies Poisson stability but is stronger; we also give results on local controllability in prescribed time that do not need this assumption on the drift vector field.

[1] Section II in [7] contains a preliminary version of some results stated here in Section 2.1.

In Section 1, we cast precisely the controllability problem we want to look at and recall the main definitions and results on the subject. Since our study is motivated by slow-fast control systems with one fast angle, we devote a few lines to this particular situation. In Section 2, we give a result of (global) controllability in arbitrary time for systems with a periodic drift together with examples and counter-examples. To our knowledge this result is original and resorts to an *ad hoc* assumption on the transportation of the controlled vector fields by the drift. This assumption is very natural in the framework of systems with one fast angular variable, and allows to prove controllability with respect to the slow ones. It turns out that the condition for these results amounts to controllability of the linear approximation along trajectories of the drift, with the same control constraint U in that linear system. Section 3, completed by a short appendix on time-varying linear control systems, addresses local controllability properties in prescribed time along one particular closed trajectory of the drift and eventually provides an alternative local-to-global proof of the results presented in Section 2.

1. Statement of the problem

1.1. **Systems.** We consider an affine control system

$$\dot{x} = X^0(x) + \sum_{k=1}^{m} u_k\, X^k(x)\,, \quad u = (u_1, \ldots, u_m) \in U \subset \mathbb{R}^m, \tag{1}$$

where the set U defines the constraints on the control and where $X^0, ..., X^m$ are $m+1$ smooth vector fields on a smooth manifold $\mathcal{M}$ of dimension d. Smooth means at least infinitely differentiable (C^∞). (When real analyticity C^ω is needed, we explicitly indicate it.)

Assumptions on the control set U. All over the paper, we assume that

$$0 \in U \text{ and } \operatorname{Span} U = \mathbb{R}^m. \tag{2}$$

The latter is not restrictive: if $\operatorname{Span} U$ was a subspace of $\mathbb{R}^m$ of dimension $m' < m$, one could pick m' new control vector fields, constant linear combinations of $X^1, \ldots, X^m$ and use them as control vector fields in a system with m' controls satisfying the assumption. The former is natural: results rely in part on assumptions on the behavior of the differential equation $\dot{x} = X^0(x)$ obtained for the zero control, that should hence be permitted.

Periodic orbits of the drift, angular variable. We assume all over section 2 and in part of section 3 the following property of the drift:

$$\begin{array}{l} X^0 \text{ has no equilibrium point, and, for any } x \text{ in } \mathcal{M}, \\ t \mapsto \exp(tX^0)(x) \text{ is periodic with minimal period } T(x). \end{array} \tag{3}$$

This defines a function $T : \mathcal{M} \to (0, +\infty)$, as smooth as the vector field X^0. The set of periodic orbits always has the structure of a smooth manifold of dimension $d-1$. Indeed, X^0 with property (3) induces the following free action of the compact Lie group $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$ on $\mathcal{M}$: $(\theta, x) \mapsto \big(\exp(\frac{1}{T(x)} 2\pi\theta\, X^0)\big)(x)$, whose orbits are the periodic orbits of X^0. The orbit space is the quotient $M = \mathcal{M}/\mathbb{S}^1$; it is a smooth manifold because the action is free and proper (free from the absence of equilibria, proper because the group is compact; see *e.g.* [6, Theorem 1.95] for the smooth manifold structure) (without using group action, the property is also clear by taking coordinates on a small transverse section of one periodic orbit as local coordinates on the quotient). This defines a smooth fibration

$$\pi : \mathcal{M} \to M \tag{4}$$

where, for each x in $\mathcal{M}$, $\pi^{-1}(\pi(x))$ is the periodic orbit of X^0 passing through x. By fibration we mean a fibre bundle over M that is, by definition, locally trivial: M can be covered with open sets O such that $\pi^{-1}(O) \simeq O \times \mathbb{S}^1$. On such an open set, denoting I the current point in M and φ in $\mathbb{S}^1$ the value in the fibre, system (1) can be written as

$$\begin{aligned} \dot{I} &= \varepsilon \sum_{k=1}^{m} u_k \, F^k(I,\varphi)\,, \\ \dot{\varphi} &= \omega(I) + \varepsilon \sum_{k=1}^{m} u_k \, f^k(I,\varphi)\,, \end{aligned} \qquad u = (u_1,\ldots,u_m) \in U\,, \tag{5}$$

for some ω, F^k and f^k whose precise definition is left to the reader. The parameter $\varepsilon > 0$ is present to indicate that the controls are small, so that the coordinates of I are "slow variables" while φ is a "fast angle", but these time scales are not used here and the reader can assume $\varepsilon = 1$. Note that $\omega(I)$ naturally comes out as $\omega(I,\varphi)$, but one may remove the dependence on φ via a change of variables $(I,\varphi) \mapsto (I, a(I,\varphi))$. In general, the fibre bundle $\mathcal{M}$ is not trivial, as illustrated by the case below that motivates our study.

Example 1. For $(q,\dot{q}) \in (\mathbb{R}^n \setminus \{0\}) \times \mathbb{R}^n$ (and $n = 2,3$), consider system (1) with

$$X^0(q,\dot{q}) = \dot{q}\frac{\partial}{\partial q} - \frac{q}{|q|^3}\frac{\partial}{\partial \dot{q}}.$$

Up to some normalisation of time, this drift accounts for the Keplerian motion of a particle in central field. As explained in [8], the dynamics of a solar sail can be described by the control-affine system (1) with this drift, plus controlled vector fields and control set U such that 0 belongs to U, the convex hull of U not being a neighborhood of the origin. We restrict to negative energies,

$$\frac{1}{2}|\dot{q}|^2 - \frac{1}{|q|} < 0,$$

and in order to have a complete drift all of whose trajectories are periodic, we get rid of collision trajectories ($q \times \dot{q} = 0$) by the standard regularization (see, *e.g.*, [14]): changing time[2] t towards s according to $\mathrm{d}t = |q|\mathrm{d}s$, the ambient manifold can be extended so that $\mathcal{M}$ be diffeomorphic to the tangent bundle of the sphere minus the zero section: $\mathcal{M} = T\mathbb{S}^n \setminus 0$. Every trajectory of the drift of the regularized system is periodic and one defines as before $\pi : \mathcal{M} \to M = \mathcal{M}/\mathbb{S}^1$. One sees that $\mathcal{M}$ is foliated by negative energy levels, and the fibre bundle is trivial if and only if the restriction of π to each energy level induces a trivial fibre bundle. Every energy level is diffeomorphic to the spherical tangent bundle of the n-dimensional sphere, denoted $S(T\mathbb{S}^n)$, and this restriction is onto the base space $N = S(T\mathbb{S}^n)/\mathbb{S}^1$. For $n = 2$, N is proved to be diffeomorphic to $\mathbb{S}^2$ [14] while one sees that $S(T\mathbb{S}^2) \simeq \mathrm{SO}(3)$. Clearly, even topologically $\mathrm{SO}(3)$ is not $\mathbb{S}^2 \times \mathbb{S}^1$ (check, *e.g.*, Poincaré polynomials), so that $\pi : \mathrm{SO}(3) \to \mathbb{S}^2$ cannot be trivial.[3] For $n = 3$, N is proved to be diffeomorphic to $\mathbb{S}^2 \times \mathbb{S}^2$ [*ibid*] while $S(T\mathbb{S}^3) = \mathbb{S}^3 \times \mathbb{S}^2$ (as a Lie group, the 3-sphere has a trivial tangent bundle); again, $\mathbb{S}^3 \times \mathbb{S}^2$ is not even homeomorphic to $\mathbb{S}^2 \times \mathbb{S}^2$ so neither the restriction nor the original fibre bundle $\pi : \mathcal{M} \to M$ can be trivial bundles.

[2]We note that condition (23) in Section 2 is insensitive to time re-parametrization under reasonable integrability properties of the change of time.

[3]The restriction is actually obtained as the Hopf fibration $\mathbb{S}^3 \to \mathbb{S}^2$, after identification of antipodal points to replace the 3-sphere by SO(3).

1.2. **Accessible sets and controllability, definitions.** A solution of (1) is a map $t \mapsto (x(t), u(t))$, defined on some interval, valued in $\mathcal{M} \times U$, where $u(.)$ is essentially bounded and $x(.)$ absolutely continuous, and that satisfies equation (1). For such a solution on an interval $[0, t_f]$, $t_f \geq 0$, one says that "the control $u(.)$ steers $x(0)$ to $x(t_f)$ in time t_f"; indeed the solution is unique given $u(.)$ and $x(0)$. One may also say that the same control steers $x(t_f)$ to $x(0)$ in time $-t_f$ for uniqueness holds also backwards.

Remark 2. In the situation where (3) is satisfied and defines π as in (4), the following fact is true: if there is a control steering an initial point x to a final point y in some time τ, there is also a control steering any point x' in the original orbit $(\pi(x') = \pi(x))$ to any point y' in the final orbit $(\pi(y') = \pi(y))$, in time less than $\tau + T(x) + T(y)$. Indeed, the new control consists in applying a zero control on $[0, t_1]$ and $[t_1 + \tau, t_1 + \tau + t_2]$ ($t_1 < T(x)$, $t_2 < T(y)$), and the original control on $[t_1, t_1 + \tau]$.

The reachable set from x_0 in time $t_f \in \mathbb{R}$ is the set of points in $\mathcal{M}$ that can be reached in forward or backward time t_f from x_0 for some control:

$$\mathcal{A}^U_{t_f}(x_0) = \{x(t_f), \text{ with } t \mapsto (x(t), u(t)) \text{ solution of (1) on } [0, t_f],\ x(0) = x_0\}\,. \tag{6}$$

It is of course implied that the interval of definition of the solution contains $[0, t_f]$. *If $t_f < 0$, $[0, t_f]$ should be understood as $[t_f, 0]$.* It is clear that:

$$y \in \mathcal{A}^U_t(x) \ \Leftrightarrow\ x \in \mathcal{A}^U_{-t}(y)\,. \tag{7}$$

We keep the set U as a superscript to stress the constraint on the control; obviously, $\mathcal{A}^V_{t_f}(x_0) \subset \mathcal{A}^U_{t_f}(x_0)$ if $V \subset U$. From this accessible set in a prescribed time, one defines $\mathcal{A}^U(x_0)$, the accessible set in arbitrary forward time, often called simply "the accessible set from x_0", although we also remain interested in properties of accessible sets in prescribed time:

$$\mathcal{A}^U(x) = \bigcup_{0 \leq t} \mathcal{A}^U_t(x)\,, \quad \mathcal{A}^U_-(x) = \bigcup_{t \leq 0} \mathcal{A}^U_t(x)\,. \tag{8}$$

Definition 3. System (1) is said to be completely controllable on $\mathcal{M}$ if $\mathcal{A}^U(x) = \mathcal{M}$ for any x in $\mathcal{M}$.

Let us also define and study a weaker controllability property where one only wants to reach all possible values of "some state variables" (say coordinates in M in the next definition) rather than all possible values of the whole state. We keep the notation (4), but this fibration is not necessarily supposed, here, to be the one induced by periodic orbits of X^0.

Definition 4. Consider System (1), and assume that a fibration (4) is defined with M a manifold of dimension $n < d$. The system is completely controllable with respect to π if $\pi\big(\mathcal{A}^U(x)\big) = M$ for any x in $\mathcal{M}$.

In the literature, this is sometimes called "partial controllability", or "output controllability", see *e.g.* [5]. When the fibration π is the one induced by periodic orbits of X^0, it means that the final orbit is prescribed, but not the position on that orbit; for systems in the form (5), the final I without prescribing φ.

Remark 5. For a general fibration (4), complete controllability implies complete controllability with respect to π. The converse does not hold: for instance, if $M = \{0\}$, controllability with respect to π holds irrespective of the control system. However, under the periodicity assumption (3) and if the fibration (4) is the one defined by the periodic orbits, complete controllability with respect to π does imply complete controllability according to Remark 2.

1.3. **Some families of vector fields.** Let $V(\mathcal{M})$ be the set of smooth (*i.e.* either C^∞ or C^ω) vector fields on $\mathcal{M}$. The Lie bracket (we refer to any advanced calculus textbook for its definition) of two elements of $V(\mathcal{M})$ also belongs to $V(\mathcal{M})$, this makes $V(\mathcal{M})$ a Lie algebra over the field $\mathbb{R}$. A family of smooth or analytic vector fields is a subset $\mathcal{F} \subset V(\mathcal{M})$; we denote by $\mathcal{F}(x)$ the subset of $T_x\mathcal{M}$ made of the values at x of vector fields in $\mathcal{F}$

$$\mathcal{F}(x) = \{X(x),\ X \in \mathcal{F}\} \subset T_x\mathcal{M}, \tag{9}$$

and by $\operatorname{Span}\mathcal{F}(x)$ the vector subspace of $T_x\mathcal{M}$ spanned by $\mathcal{F}(x)$.

Let us define the following families of vector fields, made of repeated Lie brackets of the vector fields $X^0, X^1, \dots, X^m$ defining system (1):

$$\mathcal{L} = \Big\{[X^{i_k}, [X^{i_{k-1}}, [\cdots\cdots [X^{i_2}, X^{i_1}]\cdots]]]\Big\}_{\substack{k\in\mathbb{N}\setminus\{0\}\\ (i_1,\dots,i_k)\in\{0,\dots,m\}^k}}, \tag{10}$$

$$\mathcal{L}_0 = \Big\{[X^{i_k}, [X^{i_{k-1}}, [\cdots\cdots [X^{i_2}, X^{i_1}]\cdots]]]\Big\}_{\substack{k\in\mathbb{N}\setminus\{0\}\\ i_1\in\{1,\dots,m\},\ (i_2,\dots,i_k)\in\{0,\dots,m\}^{k-1}}}, \tag{11}$$

$$\mathcal{F}_0 = \{\operatorname{ad}^j_{X^0} X^k,\ k \in \{1,\dots,m\},\ j \in \mathbb{N}\}\,. \tag{12}$$

Obviously, $\mathcal{F}_0 \subset \mathcal{L}_0 \subset \mathcal{L}$. Defining also $\operatorname{Lie}\{X^0,\dots X^m\}$ to be the smallest Lie subalgebra of $V(\mathcal{M})$ containing $X^0,\dots X^m$ and $\mathcal{I}$ the smallest Lie ideal of $\operatorname{Lie}\{X^0,\dots X^m\}$ containing $X^1,\dots X^m$. One clearly has

$$\operatorname{Span}\mathcal{L}(x) = \operatorname{Lie}\{X^0,\dots X^m\}(x)\,,\quad \operatorname{Span}\mathcal{L}_0(x) = \mathcal{I}(x)\,. \tag{13}$$

Definition 6 (Bracket generating property)**.** The control system (1), or the family of vector fields $\{X^0,\dots,X^m\}$, is called *bracket generating* at point x if and only if $\operatorname{Span}\mathcal{L}(x) = T_x\mathcal{M}$. It is called bracket generating if this is true for all x in $\mathcal{M}$.

Being "bracket generating" is also called the Lie Algebra Rank Condition (LARC) [17]. Also, System (1) is said to satisfy the *ad-condition* at x if $\mathcal{F}_0(x) = T_x\mathcal{M}$.

1.4. **Controllability when $\operatorname{conv} U$ is a neighborhood of the origin, state of the art.** It is well known that, for example according to the so-called "orbit Theorem" due to Sussmann [17], bracket generating is necessary for controllability; we will always assume that this is satisfied, explicitly or through stronger assumptions. It does not imply controllability in general but the weaker following accessibility property.

Proposition 7 (Accessibility condition, Jurdjevic-Sussmann [18])**.** *Consider system (1); assume that U satisfies condition (2) and that the vector fields are real analytic. The topological interior of $\cup_{0\le t\le t_f}\mathcal{A}^U_t(x)$ is nonempty for any x in $\mathcal{M}$ and any positive t_f if and only if System (1) is bracket generating on $\mathcal{M}$.*

The topological interior of $\mathcal{A}^U_{t_f}(x)$ is nonempty for any x in $\mathcal{M}$ and any nonzero t_f if and only if $\operatorname{Span}\mathcal{L}_0(x) = T_x\mathcal{M}$ *for all x in $\mathcal{M}$.*

In the original reference [18], as in the textbooks [1, chapter 8] or [9, chapter 3], one proves controllability by piecewise constant controls, *i.e.* analyzes the accessible sets of the family of vector fields

$$\mathcal{G} = \{\, X^0 + u_1X^1 + \cdots + u_mX^m\,,\ (u_1,\dots,u_m) \in U\} \tag{14}$$

(the accessible set of a family is the set of points that can be reached by concatenating a finte number of pieces of integral curves of vector fields in the family), rather than accessible set of the System (1). Note that $\operatorname{Lie}\mathcal{G}(x) = \operatorname{Lie}\{X^0,\dots,X^m\}$ because U satisfies (2).

One case where the assumptions of Theorem 7 also imply controllability is the one of driftless systems, namely systems (1) where X^0 is identically zero: in [13], it is proved that system (1), with X^0 assumed identically zero and U assumed to be a neighborhood of the origin, is controllable if and only if $\{X^1, \dots, X^m\}$ is bracket generating. This is however very far from the situation of (5), where the drift is assumed to be non-vanishing. It turns out that the periodicity assumption, or the more general property of Poisson stability allow one to strengthen the conclusion of Theorem 7 into controllability. Recall that, for a complete vector field X^0 on $\mathcal{M}$, a point $x \in \mathcal{M}$ is said to be (positively) *Poisson stable* for X^0 if there exists a sequence of positive times $(t_n)_n \to \infty$ such that $\exp(t_n X^0)(x) \to x$ when $n \to \infty$, and the vector field itself is said to be Poisson stable if there is a dense subset of such points. It turns out that many physical dynamical systems have this property; this makes the following result quite useful.

Theorem 8 (Bonnard, 1981, [2]). *System (1) is controllable if*

(i) *the vector field X^0 is Poisson stable,*
(ii) *the family $\{X^0, X^1, \dots, X^m\}$ is bracket generating, and*
(iii) *the convex hull of the control set U is a neighborhood of 0 in $\mathbb{R}^m$.*

It is stated precisely in this form in the recent textbook [9] (Chapter 4, Theorem 5) or in the original reference, that mentions techniques due to [10]. It has been rather widely used, for instance to prove controllability prior to solving an optimal control problem, see *e.g.* [3].

Here, we assume that (i) holds, and even the stronger periodicity assumption (3), we do assume (ii), that is anyway necessary as mentioned above, but we investigate the case when (iii) fails, zero being the boundary of U, typically the case where U is included in a non trivial convex cone with vertex at the origin, strictly convex at the origin. These positivity constraints come naturally in many physical systems, see *e.g.* [7, 8]. This situation may of course defeat controllability, as evidenced by a very simple academic example of the form (5) with $I \in M = \mathbb{R}$ and one scalar control u: $\dot{I} = \varepsilon\, u\,,\ \dot{\varphi} = 1,\ u \in U \subset \mathbb{R}$; conditions (i) and (ii) are satisfied; if $U = [-1, 1]$, (iii) is also satisfied and controllability trivially holds, while it cannot if $U = [0, 1]$ because I cannot decrease.

1.5. **Further constructions, not based on Lie brackets.** Classical controlability results are based on Lie brackets, or more precisely on the vector subspace of the tangent space at all or some points, spanned by some Lie brackets of $X^0, \dots, X^m$. When the convex hull of U is not a neighborhood of the origin, conditions based on constructing some *linear subspace* —spanned by some Lie brackets— of the tangent space at each point, and checking whether it is the whole tangent space, cannot be relevant. We present instead a construction based on convex or conic combinations, and Lie brackets have to be replaced by transporting vector fields along flows.

To the vector fields $X^0, \dots, X^m$ and the convex set U defining system (1), we associate, for any x in $\mathcal{M}$ and any real number τ, the following subset of $T_x\mathcal{M}$:

$$E^U_\tau(x) = \left\{ \sum_{k=1}^m u_k \left(\exp(-\tau X^0)_* X^k\right)(x),\ u \in U \right\} \subset T_x\mathcal{M}\,, \tag{15}$$

and, when the periodicity assumption (3) holds and defines $T(x)$,

$$\begin{aligned} E^U(x) &= \bigcup_{\tau \in [0, T(x)]} E^U_\tau(x) \\ &= \left\{ \sum_{k=1}^m u_k \left(\exp(-\tau X^0)_* X^k\right)(x),\ u \in U,\ \tau \in [0, T(x)] \right\} \subset T_x\mathcal{M}\,. \end{aligned} \tag{16}$$

Note that $E^U_\tau(x)$ is convex when U is and contains the origin, while $E^U(x)$ —or any $\bigcup_{\tau\in[t_1,t_2]} E^U_\tau(x)$ that we will sometimes use— also contains the origin but bears no special structure. In (5) coordinates, one may write (15) and (16) in a simpler form because the flow of X^0 is explicit in these coordinates $(\exp(tX^0)(I,\varphi) = (I, \varphi+\omega(I)t))$. With $x=(I,\varphi)$, one has

$$\pi'(x)\big(E^U_t(x)\big) = \left\{ \sum_{k=1}^{m} u_k\, F_k(I, \varphi+\omega(I)t),\ u\in U \right\}, \tag{17}$$

$$\pi'(x)\big(E^U(x)\big) = \left\{ \sum_{k=1}^{m} u_k\, F_k(I, \varphi)\,,\ \ (u_1,\dots,u_m)\in U\,,\ \varphi\in\mathbb{S}^1 \right\}. \tag{18}$$

Note that $E^U_t(x) = E^U_t(I,\varphi)$ depends on I but not on φ; in fact, $E^U(.)$ is invariant under the flow of X^0, and so is the function $T(.)$. These constructions are different from those based on Lie brackets, and in particular they depend both on the vector fields on on the control set U, but the *vector subspace of* $T_x\mathcal{M}$ *spanned by* the vectors in unions of $E^U_\tau(x)$, for τ in some interval of $\mathbb{R}$, can be characterized in terms of Lie brackets if all vector fields are real analytic.

Proposition 9. *If the vector fields are real analytic, one has, for any $t_1 < t_2$ and any x,*

$$\operatorname{Span} \bigcup_{\tau\in[t_1,t_2]} E^U_\tau(x) = \operatorname{Span}\mathcal{F}_0(x)\,, \tag{19}$$

with $\mathcal{F}_0$ defined in (12).

Proof. Results from Lemma 10 below applied with $\alpha_1 = t_1\pm\varepsilon$, $\alpha_2 = t_2\pm\varepsilon$, and the fact that, for any $\varepsilon>0$, $\operatorname{Span}\mathcal{F}_{(t_1-\varepsilon,t_2+\varepsilon)} \subset \bigcup_{\tau\in[t_1,t_2]} E^U_\tau(x) \subset \operatorname{Span}\mathcal{F}_{(t_1+\varepsilon,t_2-\varepsilon)}$. □

Assume that X^0 is complete and define

$$\mathcal{F}_{(\alpha_1,\alpha_2)} = \{\exp(-tX^0)_* X^k,\ k\in\{1,\dots,m\},\ t\in\mathbb{R},\ \alpha_1<t<\alpha_2\},\ \ \alpha_1<\alpha_2, \tag{20}$$

$$\mathcal{F}_\infty = \{\exp(-tX^0)_* X^k,\ k\in\{1,\dots,m\},\ t\in\mathbb{R}\}\,. \tag{21}$$

One has the following identities, with $\mathcal{F}_0$ defined by (12).

Lemma 10. *Let $\alpha_1<\alpha_2$ be some numbers. In general (*i.e. *if $X^0, X^1,\dots,X^m$ are C^∞), one has $\operatorname{Span}\mathcal{F}_0(x) \subset \operatorname{Span}\mathcal{F}_{(\alpha_1,\alpha_2)}(x) \subset \operatorname{Span}\mathcal{F}_\infty(x)$. If the vector fields are real analytic, one has $\operatorname{Span}\mathcal{F}_0(x) = \operatorname{Span}\mathcal{F}_{(\alpha_1,\alpha_2)}(x) = \operatorname{Span}\mathcal{F}_\infty(x)$.*

Proof. For x in $\mathcal{M}$ and p an element of $T^*_x\mathcal{M}$, let

$$a_k(t) = \big\langle p, \exp(-tX^0)_* X^k\,(x)\big\rangle\,,\quad t\in\mathbb{R}.$$

The map a_k is smooth and[4]

$$\frac{\mathrm{d}^j\, a_k}{\mathrm{d}t^j}(t) = \Big\langle p\,,\, \exp(-tX^0)_*\, \mathrm{ad}^j_{X^0} X^k\,(x) \Big\rangle\,. \tag{22}$$

If p belongs to the annihilator of $\operatorname{Span}\mathcal{F}_{(\alpha_1,\alpha_2)}(x)$, then $a_k(t)$ is zero for all integer k in $\{1,\dots,m\}$ and t in (α_1,α_2); differentiating j times and then taking $t=0$ and using (22) implies that $\Big\langle p, \mathrm{ad}^j_{X^0} X^k\,(x)\Big\rangle = 0$, $k\in\{1,\dots,m\}$, $j\in\mathbb{N}$, hence p must belong to the annihilator of $\operatorname{Span}\mathcal{F}_0(x)$. This proves $\operatorname{Span}\mathcal{F}_0(x) \subset \operatorname{Span}\mathcal{F}_{(\alpha_1,\alpha_2)}(x)$, while $\operatorname{Span}\mathcal{F}_{(\alpha_1,\alpha_2)}(x) \subset \operatorname{Span}\mathcal{F}_\infty(x)$ is obvious by definition. To prove the reverse inclusion, assume that p is in the annihilator of $\operatorname{Span}\mathcal{F}_0(x)$. For each integer k in $\{1,\dots,m\}$, p vanishes on all vectors $\mathrm{ad}^j_{X^0} X^k\,(x)$, $j\in\mathbb{N}$, so according to (22) $\frac{\mathrm{d}^j a_k}{\mathrm{d}t^j}(0) = 0$ for all j. Since the map a_k is real analytic when the vector fields are real analytic, it must be identically zero on $\mathbb{R}$. This proves that $\operatorname{Span}\mathcal{F}_\infty(x) \subset \operatorname{Span}\mathcal{F}_0(x)$, hence $\operatorname{Span}\mathcal{F}_0(x) = \operatorname{Span}\mathcal{F}_{(\alpha_1,\alpha_2)}(x) = \operatorname{Span}\mathcal{F}_\infty(x)$. □

[4]This is because $\frac{\mathrm{d}}{\mathrm{d}t}\left(\exp(-tY)_* Z\right)(x) = \left(\exp(-tY)_*[Y,Z]\right)(x)$ for any vector fields Y and Z.

2. Controllability in arbitrary time of systems with a periodic drift

2.1. **Result.** The main result of this section is the following theorem. It is concerned with system (1) under assumptions (3) and (2).

Theorem 11. *Assume that all vector fields are real analytic, that X^0 satisfies (3) (periodicity of all solutions), that U satisfies (2), and that*

$$\pi'(x)\left(\operatorname{conv}\left(E^U(x)\right)\right) \text{ is a neighborhood of 0 in } T_{\pi(x)}M \tag{23}$$

for all x in $\mathcal{M}$. Then, the system is completely controllable and a fortiori *completely controllable with respect to π,* i.e. $\pi\left(\mathcal{A}^U(x)\right) = M$. *Under the same conditions, controllability holds as well in backward time,* i.e. $\mathcal{A}_-^U(x) = \mathcal{M}$ *for all x in $\mathcal{M}$.*

Proof. We prove the property in forward time; the one in backward time follows upon changing each vector field X^i into $-X^i$ (this preserves all assumptions of the theorem). For a family of vector fields $\mathcal{F}$, we denote by $\mathcal{A}_\mathcal{F}(x)$ the accessible set from x of this family of vector fields in positive (unspecified) time, *i.e.* the set of points that can be reached from x by following successively the flow of a finite number of vector fields in $\mathcal{F}$, each for a certain positive time. With $\mathcal{G}$ the family of vector fields defined by (14), $\mathcal{A}_\mathcal{G}(x)$ is also the set of points that can be reached, for the control system (1), with piecewise constant controls; we are going to show that, under our assumptions, $\mathcal{A}_\mathcal{G}(x)$ is the whole manifold $\mathcal{M}$ for any x in $\mathcal{M}$; this obviously implies the Proposition. Define the families $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$ (with $\mathcal{G} \subset \mathcal{G}_1 \subset \mathcal{G}_2 \subset \mathcal{G}_3$) as follows:

$$\mathcal{G}_1 = \mathcal{G} \cup \{-X^0\}\,, \quad \mathcal{G}_2 = \{\exp(t\,X^0)_* X\,,\ X \in \mathcal{G}_1,\, t \in \mathbb{R}\}\,, \quad \mathcal{G}_3 = \operatorname{cone}\mathcal{G}_2\,, \tag{24}$$

where $\exp(t\,X^0)_* X$ denotes the push-forward of the vector field X by the diffeomorphism $\exp(t\,X^0)$ and $\operatorname{cone}\mathcal{G}_2$ denotes the family made of all vector fields that are linear combinations of the form $\sum_k \lambda_k X_k$ with each X_k in $\mathcal{G}_2$ and each λ_k a nonnegative number, $k \in \mathbb{N}$ (conical combination). For all x, one has[5]

$$\mathcal{A}_{\mathcal{G}_1}(x) = \mathcal{A}_\mathcal{G}(x) \tag{25}$$

because on the one hand our assumption on the control set implies $X^0 \in \mathcal{G}$, and on the other hand, for any $x \in \mathcal{M}$,

$$\exp(-tX^0)(x) = \exp((-t + kT(x))X^0)(x)$$

for all positive integers k, but for fixed t and x, $-t+kT(x)$ is nonnegative for k large enough. Since X^0 and $-X^0$ now belong to $\mathcal{G}_1$, we have $\exp(t\,X^0)(x) \in \mathcal{A}_\mathcal{G}(x)$ for all x in $\mathcal{M}$ and all t in $\mathbb{R}$, hence $\exp(t\,X^0)$ is according to [9, Chapter 3, Definition 5 and next lemma], a "normalizer" of the family $\mathcal{G}_1$: by virtue of Theorem 9 in the same chapter of the same reference, this implies that[5]

$$\mathcal{A}_{\mathcal{G}_2}(x) \subset \overline{\mathcal{A}_{\mathcal{G}_1}(x)} \tag{26}$$

where the overline denotes topological closure (for the natural topology on $\mathcal{M}$). Now, [1, Corollary 8.2] or [9, chapter 3, Theorem 8(b)] tell us that[5]

$$\mathcal{A}_{\mathcal{G}_3}(x) \subset \overline{\mathcal{A}_{\mathcal{G}_2}(x)}\,. \tag{27}$$

Now, (23) implies (the notation $\mathcal{G}_3(x)$ is defined in (9)) that $\mathcal{G}_3(x) = T_x\mathcal{M}$ for all x in $\mathcal{M}$, and this in turn implies that $\overline{\mathcal{A}_{\mathcal{G}_3}(x)} = \mathcal{M}$. Together with (26)-(27), this implies

$$\overline{\mathcal{A}_\mathcal{G}(x)} = \mathcal{M}\,. \tag{28}$$

[5] In the terminology of [1, Section 8.2], one could state (25), (26) and (27) as: $-X^0$ is compatible with $\mathcal{G}$, the vector fields in $\mathcal{G}_2$ are compatible with $\mathcal{G}_1$, and the vector fields in $\mathcal{G}_3$ are compatible with $\mathcal{G}_2$, respectively.

From (23) and Proposition 9, one has $\mathrm{Span}\left(\pi'(x)\,\mathcal{F}_0(x)\right)=T_{\pi(x)}M$; since $X^0(x)$ spans the tangent space to the fibre $\pi^{-1}(\pi(x))$, one has $\mathrm{Span}\left(\mathcal{F}_0(x)\cup\{X^0(x)\}\right)\subset \mathrm{Span}\,\mathcal{L}(x)$ is the whole space $T_x\mathcal{M}$, hence the system is bracket generating[6]; from [1, Theorem 8.1], (28) then implies $\overline{\mathcal{A}_{\mathcal{G}}(x)}=\mathcal{M}$. □

Comments on condition (23). According to Lemma 10, condition (23) implies the bracket generating property and in fact implies that $T_x\mathcal{M}$ is spanned by the value at x of vector fields $\mathrm{ad}^j_{X^0}X^k$, $k\in\{1,\dots,m\}$, $j\in\mathbb{N}$ only, without any brackets between vector fields, like $[X^1,X^2]$ or $[X^1,[X^0,X^1]]$. This condition amounts to the controllability of the linearized system along the periodic solution running through x, see Section 3. This condition is more difficult to check than computing Lie brackets (differentiation) and checking the rank of a family of vector fields (linear algebra), it resorts to convex optimization and is discussed in [7, 8] by the authors.

The theorem above gives a sufficient condition for global controllability in arbitrary time. No claim is made on its necessity. Let us however investigate the situation where condition (23) fails, assuming for convenience that the system is in the form (5). Assumption (23) of Theorem 11 fails if and only if, for at least one I_0 in M, there is a nonzero $p_0\in T^*_{I_0}M$ such that

$$\left\langle p_0\,,\,\sum_{k=1}^m u_k\,F^k(I_0,\varphi)\right\rangle\geq 0\,,\quad u=(u_1,\dots,u_m)\in U\,,\ \varphi\in\mathbb{S}^1\,. \tag{29}$$

Geometrically, this means that the convex cone generated by

$$\left\{\sum_{k=1}^m u_kF^k(I_0,\varphi),\ u\in U,\ \varphi\in\mathbb{S}^1\right\} \tag{30}$$

is contained in a closed half-space. Let us describe two such situations.

First, assume that for some I_0 the polar cone of the cone generated by (**??**) (*i.e.* the cone of covectors that are nonpositive evaluated on any vector in the set (**??**)) has nonempty interior, which is a reinforcement of the negation (29) of (23) ((29) only requires that that polar cone is not $\{0\}$). By continuity, the polar cone associated with I in a small enough open neighborhood O of I_0 will still has a nonempty interior. This implies that some points in a smaller open set O' cannot be reached by admissible controls without leaving O'. (short proof: fix one nonzero p in the cone at I; in coordinates, the same p will also be in the polar cone at I' for I' in O', and this implies that $\langle p,I(t)\rangle$ cannot decrease on solutions produced by controls valued in U without the solution leaving O'.) This defeats local controllability that consists in reaching all neighboring points without leaving a neighborhood; for instance the conclusions of Theorem 14 (reaching arbitrarily close points in fixed time with arbitrarily small controls) cannot hold. This does not however defeat the conclusions of Theorem 11 (complete controllability according to Definition 3 or 4) for there may be trajectories initiating from I_0 that leave O, go "far away" and reach I after re-entering O; this occurs in Example 12, System (33), showing that the converse to Theorem 11 does not hold.

A differently restrictive case is obtained when the cones generated by (**??**), not only at some point I_0 but for all I in M, are included in vector subspaces of positive codimension. This means that, for each I in M, there exists a nonzero $p\in T^*_IM$, depending on I, such that

$$\left\langle p,\,\sum_{k=1}^m u_k\,F^k(I,\varphi)\right\rangle=0\,,\quad (u_1,\dots,u_m)\in U\,,\ \varphi\in\mathbb{S}^1\,. \tag{31}$$

[6] This is the only instance in the proof where real analyticity is needed. If $X^0,\dots,X^m$ are only C^∞, the theorem still holds under the additional *assumption* that the family is bracket generating.

Although (23) cannot *a fortiori* hold in this situation, one cannot deduce from (31) any obstruction to controllability, even local, without further integrability (in the sense of Frobenius) properties. Note that U, or its convex hull, not being a neighborhood of the origin does not play any role in defeating Condition (23): indeed, if (31) holds with U spanning $\mathbb{R}^m$ by linear combinations, without sign requirement, as assumed in (2), then it also holds with $U = \mathbb{R}^m$. Example 13 below demonstrates that controllability may hold in these cases, and that this is thanks to brackets that are not present in the family $\mathcal{F}_0$ (see (12)). We already noted in Proposition 9 that Theorem 11 may yield controllability only for systems such that the brackets in $\mathcal{F}_0$ generated the whole tangent space, which is more restrictive than the general bracket generating assumption. Example 13 reflects a general family of such bracket generating systems, and Section 3 further exploits the extra assumption linked to generation by the brackets in $\mathcal{F}_0$, namely controllability of the linear approximation.

2.2. **Examples and counter-examples.** We give two academic examples, both of systems of the form (5) with state variables (I, φ).

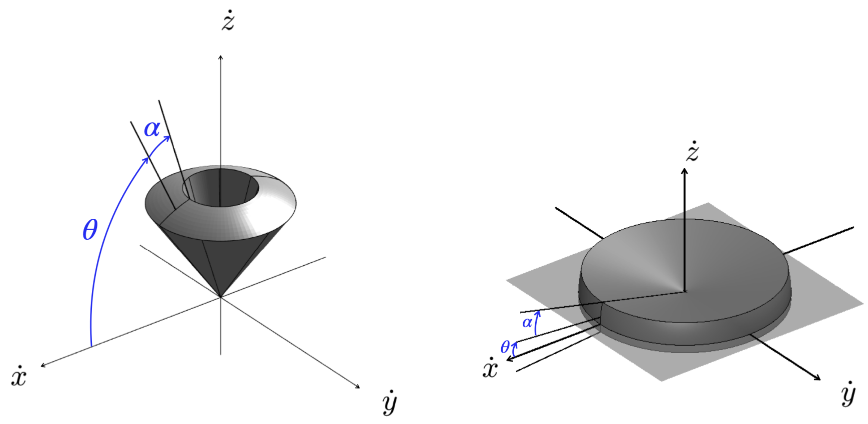
Example 12. $M = \mathbb{R}^3$, $I = (x, y, z)$. Two controls ($m = 2$).

$$\begin{aligned}
&\begin{pmatrix}\dot x\\ \dot y\\ \dot z\end{pmatrix} = \varepsilon \begin{pmatrix}\cos\varphi & -\sin\varphi & 0\\ \sin\varphi & \cos\varphi & 0\\ 0 & 0 & 1\end{pmatrix} \left(u_1 \begin{pmatrix}\cos\theta\\ 0\\ \sin\theta\end{pmatrix} + u_2 \begin{pmatrix}-\sin\theta\\ 0\\ \cos\theta\end{pmatrix}\right)\\
&\dot\varphi = 1\,,\\
&(u_1, u_2) \in U = \{(u_1, u_2) \in \mathbb{R}^2,\ u_1 \in [0, 1],\ |u_2| \le u_1 \tan\alpha\,\}\,,
\end{aligned} \tag{32}$$

where θ and α are fixed parameters, $-\pi/2 < \theta < \pi/2$, $0 < \alpha \le \pi/2$. The control set U is the 2-D cone with semi-angle α around the positive axis Ox. Let us describe the sets $E^U_\tau(I, \varphi)$ and $E^U(I, \varphi)$ defined in (15)-(16); since they are a subset of the tangent space, we take as coordinates $\dot x, \dot y, \dot z, \dot\varphi$; these sets all have a zero component on $\dot\varphi$ (*i.e.* $E^U_\tau(I, \varphi) = \pi'(I, \varphi).E^U_\tau(I, \varphi) \times \{0\}$), and depend on φ only. According to definitions, $\pi'(x, y, z, \varphi)\, E^U_\tau(x, y, z, \varphi)$ is, in the $(\dot x, \dot y, \dot z)$ space and in spherical coordinates, the 2-D convex cone (bounded by $\dot x^2 + \dot y^2 + \dot z^2 \le 1$ with semi-angle α in the meridian plane of longitude $\varphi + \tau$ around the semi-axis with longitude $\varphi + \tau$ and latitude θ; taking the union of these along τ, $\pi'(x, y, z, \varphi).E^U(x, y, z, \varphi)$ is the set of points whose radius is less than 1 and latitude is between $\theta - \alpha$ and $\theta + \alpha$. This latter set is drawn for various values of θ and a fixed value of α in Figure 1.

Clearly, Condition (23) is satisfied if $|\theta| < \alpha$, as in Figure 1-(B). If $|\theta| \ge \alpha$, as in Figure 1-(A), the condition fails and it is easy to check that controllability cannot hold because $\dot z$ is always of the sign of θ.

In the solar sail orbit transfer problem [8, 7], the role of α is played by the physical property of the solar sail itself, the more reflexivity, the larger α, while the role of θ is played by the orbital elements of the current orbit: around a given orbit, controllability occurs if that cone is open enough. It is however not correct to say that the orbital elements are fixed parameter, they are rather a part of the state: One may enrich our academic example by making θ a component of the state



(A) $|\theta| > \alpha$ (B) $|\theta| < \alpha$

FIGURE 1. This is, for System (32), a drawing of $\pi'(I,\varphi)\big(E^U(I,\varphi)\big)$ (in fact it does not depend on I, φ: say it is $\pi'(0)\big(E^U(0)\big)$) for $\alpha = \pi/20$ and two different values of θ. When $|\theta| > \alpha$ the convex hull of $\pi'(0)\big(E^U(0)\big)$ is clearly not a neighborhood of the origin as it is contained in the half space $\dot z > 0$. On the contrary, when $|\theta| < \alpha$, $\pi'(0)\big(E^U(0)\big)$ contains point with negative and positive values of $\dot z$ and its convex hull *is* neighborhood of the origin.

rather than a fixed parameter, α being still a fixed parameter:

$$
\begin{aligned}
&\begin{pmatrix}\dot x\\ \dot y\\ \dot z\end{pmatrix} = \varepsilon \begin{pmatrix}\cos\varphi & -\sin\varphi & 0\\ \sin\varphi & \cos\varphi & 0\\ 0 & 0 & 1\end{pmatrix} \left(u_1 \begin{pmatrix}\cos\theta\\ 0\\ \sin\theta\end{pmatrix} + u_2 \begin{pmatrix}-\sin\theta\\ 0\\ \cos\theta\end{pmatrix}\right)\\
&\dot\theta \;=\; u_3 \cos^2 2\theta\,,\\
&\dot\varphi \;=\; 1\\
&u \in U = \{(u_1,u_2,u_3)\in\mathbb{R}^3,\ u_1\in[0,1],\ |u_2|\le u_1\tan\alpha,\ |u_3|\le 1\}\,.\\
&(x,y,z,\theta,\varphi)\in M\times\mathbb{S}^1,\ M=\mathbb{R}^3\times\left(-\frac{\pi}{4},\frac{\pi}{4}\right)\,.
\end{aligned}
\tag{33}
$$

The interval $(-\frac{\pi}{4},\frac{\pi}{4})$ is an arbitrary choice; the factor $\cos^2 2\theta$ in $\dot\theta$ is present only to make the vector field X^3 complete, so that solutions cannot leave the state space $\mathbb{R}^3\times(-\frac{\pi}{4},\frac{\pi}{4})\times\mathbb{S}^1$ in finite time with admissible controls. If $\alpha\ge\pi/4$, condition (23) holds everywhere and the system is completely controllable by Theorem 11. If $\alpha<\pi/4$, condition (23) holds in the region $\{(x,y,x,\theta,\varphi),\ |\theta|<\alpha\}$ and fails in the region $\{(x,y,x,\theta,\varphi),\ \alpha\le|\theta|<\pi/4\}$. At points where $|\theta|>\alpha$, the strengthened version of (29) requiring that the polar cone of (29) has a nonempty interior holds, and, as previously noticed, this prevents a form of local controllability requiring that values of I close to the initial values are reached without I going far from this initial value (or, in the spirit of Section 3, that values of I close to the initial values are reached with small controls in one revolution of φ). This example is however completely controllable even if $\alpha<\pi/4$ because when starting from some $(x_0,y_0,z_0,\theta_0,\varphi_0)$ to reach $(x_f,y_f,z_f,\theta_f,\varphi_f)$, one may always use the control u_3 to reach a region where Condition (23) holds (for instance go to $(x_0,y_0,z_0,0,\varphi_0)$), then use the controls u_1,u_2 to reach x_f,y_f,z_f (possible as in (32) with $\theta=0$) with constant θ and then use again u_3 to steer θ from 0 to θ_f. In the solar sail orbit transfer problem [8, 7], one is in a similar situation: there is a limit angle above which condition (23) holds all over the set of elliptic orbits (negative energy), but if the angle α is smaller than that value, there are orbits where condition (23) holds and orbits where it fails; on orbits where it fails, it is not possible to reach all close-by orbits without "going far" first, but global controllability most probably holds there, for the same reason as here; it would be intricate (and not very interesting indeed) to show that one may reach the region where it holds from the region where it fails and vice versa, while it was very easy here on our academic example.

Example 13. Consider the following system of the form (5) with scalar control ($m=1$, $\mathcal{M}=M\times\mathbb{S}^1$, $M=\mathbb{R}^3$, $d=4$):

$$\begin{aligned}&\begin{pmatrix}\dot I_1\\ \dot I_2\\ \dot I_3\end{pmatrix}=\varepsilon\,u\Bigg(\cos\varphi\begin{pmatrix}1\\0\\-I_2/2\end{pmatrix}+\sin\varphi\begin{pmatrix}0\\1\\I_1/2\end{pmatrix}\Bigg)\,,\\ &\dot\varphi=1\,,\end{aligned}\tag{34}$$

where the scalar control is contained in a subset $U\subset\mathbb{R}$ that we do not fix yet. The system also reads $\dot x=X^0(x)+uX^1(x)$, with $x=(I_1,I_2,I_3,\varphi)$ and the vector fields X^0 and X^1 defined by:

$$X^0=\frac{\partial}{\partial\varphi}\,,\quad X^1=\cos\varphi\Big(\frac{\partial}{\partial I_1}-\tfrac12 I_2\frac{\partial}{\partial I_3}\Big)+\sin\varphi\Big(\frac{\partial}{\partial I_2}+\tfrac12 I_1\frac{\partial}{\partial I_3}\Big)\,.\tag{35}$$

Clearly, all vectors in $E^U(I,\varphi)$ have a zero component on $\partial/\partial\varphi$, *i.e.* $E^U(I,\varphi)=\pi'(I,\varphi)\left(E^U(I,\varphi)\right)\times\{0\}$, and $\pi'(I,\varphi)\left(E^U(I,\varphi)\right)$ is
- the subspace $\{\dot I_3+\frac12\big(I_2\dot I_1-I_1\dot I_2\big)=0\}$ if $U=\mathbb{R}$,
- an elliptic disc $\{\dot I_3+\frac12\big(I_2\dot I_1-I_1\dot I_2\big)=0,\ I_1^2+I_2^2\le 1\}$ in that plane if $U=[0,1]$ or $U=[-1,1]$.
As a result, Condition (23) fails whether U is equal to $[0,1]$, $[-1,1]$ or even $\mathbb{R}$.

Let us situate this example with respect to the brief review of the ways condition (23) may fail (end of section 2.1). Obviously, (31) is valid with $p=(I_2,-I_1,2)$. No strengthening of inequality (29) may hold: when fixing I and p and varying φ, either $\langle p,X^0(I,\varphi)\rangle$ is identically zero or it strictly changes sign.

Controllability holds if $U=[-1,1]$, and a fortiori $U=\mathbb{R}$, from Theorem 8 because the solutions of the drift are periodic and the values of the vector fields

$$X^0\,,\quad X^1\,,\quad [X^0,X^1]\,,\quad [\,X^1,[X^0,X^1]\,]\tag{36}$$

span the whole tangent space, hence the system is bracket generating. This illustrates the fact that our sufficient condition for controllability of (1) captures only the cases where, if U was all $\mathbb{R}^m$, the family of vector fields $\mathcal{F}_0$, complemented by X^0, would be generating, which misses all the brackets where control vector fields appear more than once, like $[X^1,[X^0,X^1]]$ above, and is hence more restrictive than the "bracket generating" property. Controllability also holds if $U=[0,1]$, this is explained in Section 3.2, in Example 20, that is a continuation of the present one.

3. Local controllability in prescribed time

3.1. **First order controllability along one trajectory of the drift.** In this subsection, we assume that

$$t\mapsto\bar x(t)\text{ is a solution of }\dot x=X^0(x),\ \text{defined on the time-interval }[0,t_f]\,.$$

We allow $t_f \in \mathbb{R}$ to be negative, and $[0, t_f]$ should be understood as $[t_f, 0]$ if $t_f < 0$. We define the following subset of $T_{\bar{x}(0)}\mathcal{M}$, attached to the solution $\bar{x}(.)$:

$$\bar{E} = \operatorname{conv}\left(\bigcup_{t\in[0,t_f]} E_t^U(\bar{x}(0)) \right) \subset T_{\bar{x}(0)}\mathcal{M}\,, \tag{37}$$

where $E_t^U(.)$ was defined in (15). Next theorem characterizes local controllability along the solution $\bar{x}(.)$ in terms of $\bar{E}$; it is stated in two parts, where the second part deals with partial controllability with respect to a projection π, a local version of the property in Definition 4. Part I is formally a particular case of Part II with $M = \mathcal{M}$ and $\pi = \mathrm{Id}$; we state it independently for readability.

Theorem 14. *Consider system (1) with U a convex compact subset of $\mathbb{R}^m$ containing the origin, and $\bar{x}(.)$ and $\bar{E}$ be as described above.*

I. *If*

$$\bar{E} \textit{ is a neighborhood of 0 in } T_{\bar{x}(0)}\mathcal{M}\,, \tag{38}$$

then $\mathcal{A}_{t_f}^{\varepsilon U}(\bar{x}(0))$ is a neighborhood of $\bar{x}(t_f)$ in $\mathcal{M}$ for any $\varepsilon > 0$.

II. *Let $\pi : \mathcal{M} \to M$ be a smooth fibration[7] with M a manifold of dimension $n \le d$. If*

$$\big(\pi \circ \exp(tX^0)\big)'(\bar{x}(0))\,\big(\bar{E}\big) \;\; \textit{is a neighborhood of 0 in } T_{\pi(\bar{x}(t_f))}M\,, \tag{39}$$

then $\pi\Big(\mathcal{A}_{t_f}^{\varepsilon U}(\bar{x}(0))\Big)$ is a neighborhood of $\pi(\bar{x}(t_f))$ in M for any $\varepsilon > 0$.

Remark 15 (local controllability). Let us explain why this is a local controllability result around the solution $t \mapsto (\bar{x}(t), 0)$. Clearly, taking ε small enough makes the control arbitrarily small and the solutions that provide controllability arbitrarily close to the reference one. In particular, this property is true for our system if and only if it is true for any system $\dot{x} = f(x, u)$ such that $f(x, u)$ and $X^0 + \sum_k u_k\, X^k(x)$ coincide on some neighborhood of $\bar{x}([0, t_f]) \times \{0\}$, hence it is a local property around that solution in the sense that it only depends on the germ of the control system on the locus of this solution.

Remark 16 (assumptions on U). This section technically needs that U be compact and convex. If there is a compact convex subset of U such that (38) holds also when replacing U with it, one may obviously apply the theorem with that subset instead of U. Convexity is an assumption, not made in Section 2, that would probably have to be relaxed.

Let us investigate Condition (38), and its relation with controllability of the control-constrained linear approximation of the control system (1) along the solution $\bar{x}(.)$. This linear approximation is the linear time-varying system whose solutions give the first order variation of the solutions with respect to variations of the control; it can be written as

$$\begin{aligned} \dot{\xi} &= \sum_{k=1}^{m} \delta u_k\, \big(\exp(-tX^0)_* X^k\big)(\bar{x}(0))\,, \\ \delta x(t) &= \exp(tX^0)'(\bar{x}(0))\, \xi(t)\,, \end{aligned} \tag{40}$$

with initial condition $\xi(0) = 0$, where $\delta u = (\delta u_1, \ldots, \delta u_m)$ is a small variation of the control around zero and $\delta x(t)$ the corresponding small variation of the state around $\bar{x}(t)$; ξ represents the small variations of $y(t) = \exp(-tX^0)(x(t))$; it is more efficient and more intrinsic to write the differential equation for ξ than for δx

[7]For the present purpose, a submersion from a neighborhood of $\bar{x}(t_f)$ onto an open subset of $\mathbb{R}^n$ would be enough for everything is local around $\bar{x}(t_f)$ and $\pi(\bar{x}(t_f))$.

(writing $\dot{\delta x} = \frac{\partial X^0}{\partial x}(\bar{x}(t))\,\delta x + \sum_1^m \delta u_k\, X^k(\bar{x}(t))$ makes sense in coordinates, but (40) is the best intrinsic translation of it for ξ lives in the same tangent space for all t, contrary to δx).

Although it is not common (strictly speaking, a linear system has a linear space as state and control space), we may also constrain the control $\delta u = (\delta u_1, \ldots, \delta u_m)$ of the linear system (40) to some subset V of $\mathbb{R}^m$, and define, for this linear control-constrained time-varying system, the accessible set at time $t > 0$ from x_0 at time zero like we did in (6)-(8) for nonlinear time-invariant systems; the accessible set at time t from δx_0 at time 0 for the time-varying linear system (40) with constraint $\delta u \in V$ is the following subset of $T_{\bar{x}(t)}\mathcal{M}$:

$$\mathsf{A}^V_{0,t}(\delta x_0) = \{\delta x(t),\ \text{with } s \mapsto (\xi(s), \delta u(s), \delta x(s)) \text{ solution of (40) on } [0,t], \\ \delta x(0) = \delta x_0,\ \delta u(s) \in V,\ \text{a.e. } s \in [0,t]\}\,. \tag{41}$$

This is consistent with (52) in Appendix A. The following result states that condition (38) is equivalent to some local controllability of the linear approximation.

Lemma 17. **I.** *The linear constrained attainable set $\mathsf{A}^U_{0,t_f}(0)$ is a neighborhood of the origin in $T_{\bar{x}(t_f)}\mathcal{M}$ if and only if Condition (38) holds.*

II. *The set $\pi'(\bar{x}(t_f))\big(\mathsf{A}^U_{0,t_f}(0)\big)$ is a neighborhood of the origin in $T_{\pi(\bar{x}(t_f))}M$ if and only if Condition (39) holds.*

Proof. This follows from Proposition 21 for point I and Proposition 22 for point II (Condition (ii)) in the Appendix, applied with $t_0 = 0$, with (40) playing the role of (54). In coordinates, the k^{th} column of $B(s)$ is indeed the coordinate vector of $X^k(\bar{x}(s))$, and the k^{th} column of $\phi(0,s)B(s)$ is the coordinate vector of $\big(\exp(-sX^0)_* X^k\big)\,(\bar{x}(0))$. □

Let us now give the proof of Theorem 14. As we already stated, it would be very classical if we assumed U to be a neighborhood of the origin because that would allow to use simply the implicit function theorem for the end point mapping from $\bar{x}(0)$ at the zero control.

Proof of Theorem 14. We give the proof of point II. The reader seeking a proof of point I (anyway a particular case of point II) without reference to the projection π should simply replace π with Id, M with $\mathcal{M}$, n with d, $\big(\pi \circ \exp(tX^0)\big)'(\bar{x}(0))\,\big(\bar{E}\big)$ with $\bar{E}$, and $\big(\pi \circ \exp(tX^0)\big)'(\bar{x}(0))\,\big(\mathsf{A}^U_{t_f}(0)\big)$ with $\mathsf{A}^U_{t_f}(0)$.

According to Lemma 17 (point II), Condition (38) implies that the projection $\pi'(\bar{x}(t_f))\big(\mathsf{A}^U_{t_f}(0)\big)$ of the accessible set of (40) in time t_f is a neighborhood of the origin in $T_{\bar{x}(t_f)}\mathcal{M}$. Let $e_0, \ldots, e_n$ be the vertices of a convex polyhedron that is a neighborhood of the origin contained in $\pi'(\bar{x}(t_f))\,\Big(\mathsf{A}^U_{t_f}(0)\Big)$. Since 0 is in the interior of the polyhedron generated by $e_0, \ldots, e_n$, there exist $\lambda^0_0, \ldots, \lambda^0_d$ with

$$\sum_{i=0}^{n} \lambda^0_i e_i = 0,\ \ \sum_{i=0}^{n} \lambda^0_i = 1, \qquad \text{and} \qquad \lambda^0_i > 0,\ i \in \{0, \ldots, n\}\,. \tag{42}$$

Since $e_0, \ldots, e_n$ are all in $\mathsf{A}^U_{t_f}(0)$, there also exist $\mathfrak{u}_0, \ldots, \mathfrak{u}_n$, some integrable U-valued controls, defined on $[0, t_f]$, that steer the origin to $e_0, \ldots, e_n$ respectively, in time t_f, for the linear time-varying system (40). Since U is convex, the control $t \mapsto u^{\text{zer}}(t) = \sum_0^n \lambda^0_i \mathfrak{u}_i(t)$ takes values in U's topological interior; by linearity, it steers the origin to the origin, for the linear system (40). For any positive ε no larger than 1 (of course unrelated to the ε in (5)), still by linearity, the controls $\varepsilon\mathfrak{u}_0, \ldots, \varepsilon\mathfrak{u}_n$ steer the origin to points with projection $\varepsilon e_0, \ldots, \varepsilon e_n$ respectively, and the control $\varepsilon u^{\text{zer}}$ takes values in εU's topological interior and steers the origin to the origin.

Now going back to the nonlinear system (1), let $\mathcal{E}:\mathrm{L}^\infty([0,t_f],\mathbb{R}^m)\to\mathcal{M}$ be the end-point mapping at time t_f starting from $\bar{x}(0)$ for the nonlinear system (1). (To avoid problems due to non-complete vector fields, they may all be multiplied by some smooth cut-off functions equal to 1 in a neighborhood of $\bar{x}([0,t_f])$ to ensure that solutions do not explode before t_f, without changing the local property in this neighborhood.) All we need to prove is that the image by $\pi\circ\mathcal{E}$ of controls valued in εU is a neighborhood of $\pi(\bar{x}(t_f))$ for any $\varepsilon>0$.[8] It is well known that $\mathcal{E}$ is continuously differentiable and that its derivative at some control is the end-point mapping, with initial point the origin, of the linearized control system (40) at this control. Consider the continuously differentiable map $F:\mathbb{R}^d\times\mathbb{R}\to\mathcal{M}$ defined by

$$F(\mu_1,\ldots,\mu_d,\varepsilon)=\pi\circ\mathcal{E}\left(\varepsilon\, u^{\mathrm{zer}}+\sum_{i=1}^{d}\mu_i(\mathfrak{u}_i-\mathfrak{u}_0)\right). \tag{43}$$

Since $\mathcal{E}$ maps the zero control to $\bar{x}(t_f)$, and $\frac{\partial F}{\partial\varepsilon}(0,\ldots,0,0)$ and $\frac{\partial F}{\partial\mu_i}(0,\ldots,0,0)$, $i\in\{1,\ldots,n\}$, are the projections of the images by the end-point mapping of the linear system (40) of u^{zer} and $\mathfrak{u}_i-\mathfrak{u}_0$ respectively, these images being zero and e_i-e_0 respectively, one has

$$\begin{aligned}&F(0,\ldots,0,0)=\pi(\bar{x}(t_f))\,,\quad \frac{\partial F}{\partial\varepsilon}(0,\ldots,0,0)=0\,,\\ \text{and}\quad&\frac{\partial F}{\partial\mu_i}(0,\ldots,0,0)=e_i-e_0\,,\quad i\in\{1,\ldots,n\}\,.\end{aligned} \tag{44}$$

The third relation in (44) implies that the Jacobian of F with respect to $\mu_1,\ldots,\mu_n$ at $(0,\ldots,0,0)$ is invertible (the vectors $e_1-e_0,\ldots,e_n-e_0$ are linearly independent because $e_0,e_1,\ldots,e_n$ are affinely independent); then, from the first and third relations, the inverse function Theorem yields a smooth map $\varepsilon\mapsto(\mu_1(\varepsilon),\ldots,\mu_n(\varepsilon))$ such that

$$F(\mu_1(\varepsilon),\ldots,\mu_n(\varepsilon),\varepsilon)\equiv\pi(\bar{x}(t_f))\,. \tag{45}$$

The second relation in (44) implies that the derivative of that map with respect to ε at zero is zero, hence $|\mu_i(\varepsilon)|<C\varepsilon^2$ for some $C>0$. By continuity, for ε small enough, the Jacobian of F with respect to $\mu_1,\ldots,\mu_n$ at $(\mu_1(\varepsilon),\ldots,\mu_n(\varepsilon),\varepsilon)$ is also invertible, hence the map $(\mu_1,\ldots,\mu_d)\mapsto F(\mu_1,\ldots,\mu_n,\varepsilon)$ is open at $(\mu_1(\varepsilon),\ldots,\mu_n(\varepsilon))$ for these small values of ε, meaning that:

$$\left.\begin{array}{l}\text{if } \varepsilon>0 \text{ is small enough, } F(\Omega,\varepsilon) \text{ is a neighborhood of } \bar{x}(t_f)\\ \text{for any neighborhood } \Omega \text{ of } (\mu_1(\varepsilon),\ldots,\mu_n(\varepsilon)).\end{array}\right\} \tag{46}$$

Defining $\lambda_0(.),\ldots,\lambda_n(.)$ and the control $u^\varepsilon:[t_0,t_f]\to\mathbb{R}^m$ by

$$\begin{aligned}&\lambda_0(\varepsilon)=\varepsilon\lambda_0^0-\sum_{j=1}^{n}\mu_j(\varepsilon)\,,\qquad \lambda_i(\varepsilon)=\varepsilon\lambda_i^0+\mu_i(\varepsilon)\,,\ 1\le i\le n\,,\\ &\text{and}\quad u^\varepsilon(t)=\sum_{k=0}^{n}\lambda_k(\varepsilon)\mathfrak{u}_k(t)\,,\end{aligned}$$

[8] In fact, this could be proved using the a conic open mapping result, like the one stated in [1, Lemma 12.4], to the map $(\lambda_0,\ldots,\lambda_n)\mapsto\pi\circ\mathcal{E}(\sum_{i=1}^n\lambda_i\mathfrak{u}_i)$: our constructions imply that the derivative at zero of this map sends the positive orthant of $\mathbb{R}^{n+1}$ onto $\mathbb{R}^n$, and, according to the Lemma we refer to, this implies that the map itself sends any neighborhood of the origin in that positive orthant onto a neighborhood of $\bar{x}(t_f)$ in $\mathcal{M}$. We keep a proof for self-containedness and because the construction of a family of controls valued in the interior of U yielding the same initial and finaly points as $\bar{x}(.)$ is more constructive; they are obtained through the implicit function theorem, taking advantage of smoothness of the map, only required to be Lipschitz continuous in the above mentionned open mapping results.

equation (43) reads $F(\mu_1(\varepsilon),\dots,\mu_n(\varepsilon),\varepsilon)=\mathcal{E}(u^\varepsilon)$. The numbers $\lambda_i(\varepsilon)$, $0\le i\le n$ are all positive if ε is small enough because $\lambda_i^0>0$, and we proved that $|\mu_i(\varepsilon)|<C\varepsilon^2$; they also satisfy $\sum_0^n\lambda_k(\varepsilon)=\varepsilon$; this implies that u^ε takes values in the interior of εU if ε is small enough. Taking ε small enough that the above holds and Ω a small enough neighborhood of $(\mu_1(\varepsilon),\dots,\mu_n(\varepsilon))$ that the control $\varepsilon\, u^{\mathrm{zer}}+\sum_{i=1}^n\mu_i(\mathfrak{u}_i-\mathfrak{u}_0)$ takes values in εU if $(\mu_1,\dots,\mu_d)$ is in Ω, Property (46) implies that the image by $\pi\circ\mathcal{E}$ of a set of controls valued in εU cover a neighborhood of $\pi(\bar{x}(t_f))$, proving the theorem for small enough values of ε, hence also for large values because $\mathcal{A}^{\varepsilon U}_{t_f}(\bar{x}(0))$ increases with ε. (Since U is convex and contains 0, one has $\varepsilon U\subset\varepsilon' U$ if $\varepsilon<\varepsilon'$.) □

3.2. **Consequences in the case of a periodic drift.** Let us now examine the results from Theorem 14 in the case studied in Section 2 where all solutions of the drift are periodic (condition (3)). Applying Theorem 14 with the periodic solutions as reference solution $\bar{x}(.)$, yields, under the hypothesis of Theorem 11, the following property of local controllability "over one period" by small controls.

Theorem 18. *Assume that the vector field X^0 satisfies the periodicity assumption (3), and that U is convex and satisfies conditions (2).*

- **I.** *If* $\operatorname{conv} E^U(x)$ *is a neighborhood of the origin in $T_x\mathcal{M}$, then $\mathcal{A}^{\varepsilon U}_{T(x)}$ is a neighborhood of x in $\mathcal{M}$ for all $\varepsilon>0$.*
  The same property holds for $\mathcal{A}^{\varepsilon U}_{-T(x)}$ (backward time).
- **II.** *Let π be the projection defined in (4), with M the set of periodic orbits. If* $\pi'(x)\big(\operatorname{conv} E^U(x)\big)$ *is a neighborhood of the origin in $T_{\pi(x)}M$, then $\pi\left(\mathcal{A}^{\varepsilon U}_{T(x)}(x)\right)$ is a neighborhood of $\pi(x)$ in M for all $\varepsilon>0$.*
  The same property holds for $\pi\Big(\mathcal{A}^{\varepsilon U}_{-T(x)}\Big)$ (backward time).

Proof. Parts I and II in this theorem are consequences of parts I and II in Theorem 14, with $t_f=T(x)$, $\bar{x}(0)=\bar{x}(t_f)=x$. □

Example 19 (Continuation of Example 12). As a complement to the global controllability properties displayed before, Theorem 18 yields local controllability "over one period" for System (33) at points (x,y,z,θ,φ) such that $|\theta|<\alpha$, or at any point (x,y,z,φ) for System (32) if the fixed parameters θ,α satisfy $|\theta|<\alpha$.

Note that this theorem is indeed a local controllability result around the considered periodic solution, as explained in Remark 15. It is also exactly the local property that we negated at the end of Section 2.1 (the second case where the normal cone to $\operatorname{conv}(E^U(s)$ has a nonempty interior).

Before discussing an example where the conditions of Theorem 18 fail (namely $\pi'(x)\big(\operatorname{conv} E^U(x)\big)$ is a not neighborhood of the origin in $T_{\pi(x)}M$), but controllability follows from other considerations, let us see how one may recover Theorem 11 as a consequence of Theorem 14 via a "local to global" proof. The statement is the same, we simply state the alternative proof.

Alternative proof of Theorem 11 (under the assumption that U is convex).
According to Theorem 18, part II, $\pi\big(\mathcal{A}^U_{T(y)}(y)\big)$ is a neighborhood of $\pi(y)$ in M for any y in $\mathcal{M}$, and so is $\pi\big(\mathcal{A}^U_{-T(y)}(y)\big)$. For x in M, let us show that $\mathcal{A}^U(x)$ is both open and closed in M; this implies the theorem by connectedness of M. First, consider y in $\mathcal{A}^U(x)$, so that $\mathcal{A}^U(x)$ contains $\mathcal{A}^U_{T(y)}(y)$, and $\pi\big(\mathcal{A}^U(x)\big)$ contains $\pi\big(\mathcal{A}^U_{T(y)}(y)\big)$, that was just pointed out as a neighborhood of $\pi(y)$, hence $\pi\big(\mathcal{A}^U(x)\big)$ is a neighborhood of $\pi(y)$, and, using Remark 2 (Section 1.1), $\mathcal{A}^U(x)=\pi^{-1}\big(\pi\big(\mathcal{A}^U(x)\big)\big)$ is a neighborhood of y; openness is proven. Now suppose that some y is in $\overline{\mathcal{A}^U(x)}$, hence $\pi(y)$ in $\overline{\pi\big(\mathcal{A}^U(x)\big)}$. We pointed out that $\pi\big(\mathcal{A}^U_{-T(y)}(y)\big)$ is a

neighborhood of $\pi(y)$; hence it intersects $\pi\big(\mathcal{A}^U(x)\big)$, *i.e.* there is some z such that $z\in\mathcal{A}^U_{-T(y)}(y)$ and $z'\in\mathcal{A}^U(x)$ such that $\pi(z)=\pi(z')$, but, according to Remark 2 again, $z'\in\mathcal{A}^U(x)$ then implies $z\in\mathcal{A}^U(x)$; $z\in\mathcal{A}^U(x)$ and $y\in\mathcal{A}^U_{T(y)}(z)$ do imply $y\in\mathcal{A}^U(x)$; this proves closedness and ends the proof of the theorem. □

Example 20 (Continuation of Example 13). Let us come back to System (34). We saw that it does not satisfy the conditions of Theorem 11 or 18, whether U is $[-1,1]$ or $[0,1]$, while complete controllability (Definition 3) holds if U is $[-1,1]$, for other reasons than Theorem 11. We now examine this same system from the point of view of the conclusions of Theorem 18 (that cannot be applied), i.e. local controllability over one period, when U is $[-1,1]$ or $[0,1]$, the latter case being more in the spirit of the present paper. We are going to prove the following three properties for System (34) (recall that the state x is (I,φ), with $I\in\mathbb{R}^3$, $\pi(I,\varphi)=I$, and the period $T(x)$ is $T(I)=2\pi$):

(a) $\pi\big(\mathcal{A}^{[-\varepsilon,\varepsilon]}_{T(I)}(I,\varphi)\big)$ is a neighborhood of I,
(b) $\pi\big(\mathcal{A}^{[0,\varepsilon]}_{T(I)}(I,\varphi)\big)$ *is not* a neighborhood of I,
(c) $\pi\big(\mathcal{A}^{[0,\varepsilon]}_{2\,T(I)}(I,\varphi)\big)$ is a neighborhood of I,

for any ε, $0<\varepsilon\le 1$. Following the alternative proof of Theorem 11 above, Points (a) and (c) imply "complete controllability with respect to π" if $U=[-1,1]$ (as already seen at Example 13) and $U=[0,1]$ (which is new). These points are outside the scope of the results of the paper, they are presented to investigate, on an example, controllability properties under weaker assumptions. Before establishing them, note that the general solution of (34) starting from $(I_1^0,I_2^0,I_3^0,\varphi^0)$ at time zero is given by $\varphi(t)=\varphi^0+t$ and, with complex notations for (I_1,I_2),

$$\begin{aligned} I_1(t)+i\,I_2(t) &= I_1^0+i\,I_2^0+\int_0^t u(s)\,e^{i\,(\varphi^0+s)}\mathrm{d}s\,,\\ I_3(t) &= I_3^0+\frac12\iint_{0\le r\le s\le t} u(s)\,u(r)\sin(s-r)\,\mathrm{d}r\,\mathrm{d}s\\ &\qquad+\frac12\,\Im\Big(\big(I_1^0-i\,I_2^0\big)\,\big(I_1(t)+i\,I_2(t)\big)\Big)\,, \end{aligned}\tag{47}$$

The very last term is zero if $(I_1(t),I_2(t))=(I_1^0,I_2^0)$. For intuition behind equations, recall that, for this system, since $\dot I_3=\frac12(I_1\dot I_2-I_2\dot I_1)$, the variation $I_3(t_2)-I_3(t_1)$ is the area swept from the origin by the plane curve $t\mapsto(I_1(t),I_2(t))$ on the interval $[t_1,t_2]$; φ is (if u is positive, $\varphi+\pi$ if negative) the polar angle of the velocity vector on that planar curve; since it increases with time ($\varphi(t)=\varphi^0+t$), this parameterized planar curve is always "turning left" when u is nonzero, with possible cusps or non-smooth points if u vanishes. The following property states that the area swept from the origin by a closed curve $(I_1(.),I_2(.))$ generated by a non-negative control over a time interval of length at most 2π cannot be negative, which is a clear obstruction to local controlability over one period and implies point (b):

$$\Big(u(.)\ge 0 \text{ and } I_1(2\pi)=I_1^0 \text{ and } I_2(2\pi)=I_2^0\Big)\quad\Rightarrow\quad I_3(2\pi)\ge I_3^0\,.\tag{48}$$

To prove this, first decompose the integral in the expression for $I_3(2\pi)$ in (47):

$$\textstyle\iint_{0\le r\le s\le 2\pi}=\iint_{0\le r\le s\le\pi}+\iint_{\pi\le r\le s\le 2\pi}+\iint_{(s,r)\in[\pi,2\pi]\times[0,\pi]}\,.\tag{49}$$

From (47), $(I_1(2\pi),I_2(2\pi))=(I_1^0,I_2^0)$ implies $\int_0^{2\pi}u(r)\sin(s-r)\mathrm{d}s=0$ for any s, hence $\iint_{(s,r)\in[\pi,2\pi]\times[0,2\pi]}u(s)\,u(r)\sin(s-r)\mathrm{d}r\mathrm{d}s=0$, implying that the last term in (49) is equal to

$$\textstyle -\iint_{(s,r)\in[\pi,2\pi]\times[\pi,2\pi]}\sin(s-r)\,u(s)\,u(r)\,\mathrm{d}s\,\mathrm{d}r,$$

but this is zero by symmetry around the axis $r = s$; hence the integral giving $I_3(2\pi)$ in (47) reduces to the region $\{0 \le r \le s \le \pi\} \cup \{\pi \le r \le s \le 2\pi\}$, where $u(s)\,u(r)\,\sin(s-r)$ is non negative; this proves (48), hence Point (b).

Let us now turn to points (a) and (c), that in particular imply that the above construction no longer holds if either the control can change sign or the time interval is longer. To prove these, we will follow the "return method" described in [4], that consists in constructing loops around any point, along which the linear approximation is controllable; this is also similar to the proof of Theorem 18, except the construction is ad'hoc to this example, and controllability of the linear approximation is obtained by direct computation. For Point (a), consider the piecewise

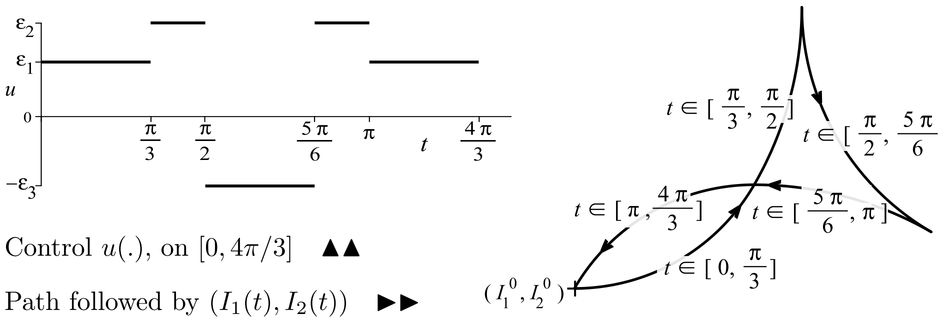


FIGURE 2. Left: the control used for Point (a) in Example 20; the constants $\varepsilon_1, \varepsilon_2, \varepsilon_3$ are tuned according to (50).
Right: the closed curve $t \mapsto (I_1(t), I_2(t))$ produced by this control on the interval $[0, \frac{4\pi}{3}]$. It is assumed that $\varphi(0) = 0$; if not, the figure on the right is rotated around (I_1^0, I_2^0) by an angle $\varphi(0)$.

constant control depicted in Figure 2 (left) for t in $[0, \frac{4\pi}{3}]$, continued by zero on $[\frac{4\pi}{3}, 2\pi]$. The constants $\varepsilon_1, \varepsilon_2, \varepsilon_3$ are related by

$$\varepsilon_1 = \sqrt{\frac{(2\pi+3)\sqrt{3} - 5\pi}{2\pi - 3\sqrt{3}}}\,\varepsilon_2\,, \quad \varepsilon_3 = (\sqrt{3}-1)\,\varepsilon_2\,. \tag{50}$$

Choosing $\varepsilon_2 < \varepsilon$ and applying (50) implies $0 < \varepsilon_1 < \varepsilon_3 < \varepsilon_2 < \varepsilon$, hence a control valued in the interior of $[-\varepsilon, \varepsilon]$. The second relation is needed to have $(I_1(\frac{4\pi}{3}), I_2(\frac{4\pi}{3})) = (I_1^0, I_2^0)$, *i.e.* a closed curve in the I_1, I_2 plane; the first relation implies that $I_3(\frac{4\pi}{3}) = I_3^0$ (or that, for the 8-shaped closed planar curve displayed in Figure 2, that (more or less) resembles a goldfish, the area of the "body", counted positively because run counter clockwise, is equal to the are of the "tail", counted negatively because run clockwise). To prove that the linear approximation along this loop is controllable, it is sufficient to prove it on any subinterval. Take for example a subinterval of $[0, \frac{\pi}{3}]$, where the control is constant equal to ε_1; the linear approximation reads $\dot{\delta I} = A(t)\delta I + B(t)\delta u$, with

$$A(t) = \varepsilon_1 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\sin\varphi(t) & \cos\varphi(t) & 0 \end{pmatrix} \text{ and } B(t) = \varepsilon_1 \begin{pmatrix} \cos\varphi(t) \\ \sin\varphi(t) \\ I_1(t)\sin\varphi(t) - I_2(t)\cos\varphi(t) \end{pmatrix}.$$

The vectors $B(t), (\frac{\mathrm{d}}{\mathrm{d}t} - A(t))B(t), (\frac{\mathrm{d}}{\mathrm{d}t} - A(t))^2 B(t)$ are always a basis of $\mathbb{R}^3$, as seen in the following formulas, where the t argument is omitted in the right-hand sides:

$$(\tfrac{\mathrm{d}}{\mathrm{d}t} - A(t))B(t) = \varepsilon_1^2 \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ I_1\cos\varphi + I_2\sin\varphi \end{pmatrix}, \ (\tfrac{\mathrm{d}}{\mathrm{d}t} - A(t))^2 B(t) = -\varepsilon_1^2 B(t) + \begin{pmatrix} 0 \\ 0 \\ 2\,\varepsilon_1^3 \end{pmatrix}.$$

Controllability of the linear approximation follows, according to [16] (see also our remarks after Equation (56).

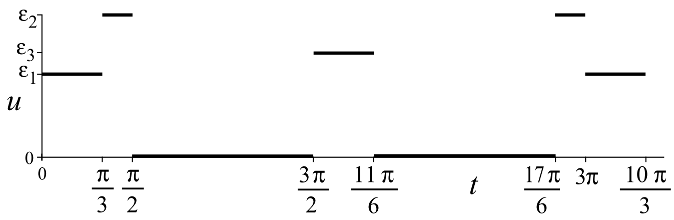


FIGURE 3. Control used for Point (c), Example 20. It produces the same closed curve as in Figure 2, except it has one "stop" of length π at each cusp (intervals $[\pi/2, 3\pi/2]$ and $[11\pi/6, 17\pi/6]$).

Let us now turn to Point (c). Instead of the control $-\varepsilon_3$ on the interval $[\frac{\pi}{2}, \frac{5\pi}{6}]$ (see Figure 2, left), that can no longer be used, one may use a zero control on $[\frac{\pi}{2}, \frac{3\pi}{2}]$, causing (I_1, I_2) to rest while φ increases from $\varphi(\frac{\pi}{2})$ to $\pi + \varphi(\frac{\pi}{2})$, and apply ε_3 on the time interval $[\frac{3\pi}{2}, \frac{11\pi}{6}]$, producing the same arc in the plane I_1, I_2 as $-\varepsilon_3$ on the time interval $[\frac{\pi}{2}, \frac{5\pi}{6}]$. A zero control on another time-interval of length π, after time $11\pi/6$, will restore the right orientation for the next (positive) controls. In the end, the control $[0, 10\pi/6] \to [0, 1]$ depicted in Figure 3 produces the same closed curve $t \mapsto (I_1(t), I_2(t))$ pictured in Figure 2, except there is a "stop" of length π at each cusp, the parameterization by t being shifted accordingly. Controllability of the linear approximation follows in the same way as for Point (a).

APPENDIX A. CONTROLLABILITY OF TIME-VARYING LINEAR SYSTEMS WITH CONSTRAINED CONTROL

Consider a time-varying linear control system, with constraints on the control:

$$\dot{z} = A(t)z + B(t)v\,, \quad v \in V\,, \tag{51}$$

where the state z belongs to $\mathbb{R}^d$, V is a compact subset of $\mathbb{R}^m$, and $t \mapsto A(t)$ and $t \mapsto B(t)$ are smooth maps $\mathbb{R} \to \mathbb{R}^{d\times d}$ and $\mathbb{R} \to \mathbb{R}^{d\times m}$ respectively. This appendix is devoted to giving properties of the set

$$\mathsf{A}^V_{t_0,t_f}(z_0) = \{z(t_f),\ \text{with } t \mapsto (z(t), v(t)) \text{ solution of (51)},\ z(t_0) = z_0\} \tag{52}$$

of points that can be reached at time t_f starting from z_0 at time t_0. It depends on both t_0 and t_f, rather than on $t_f - t_0$ only for a time-invariant system. It is known to be convex, at least if V is compact, whether V is convex or not, see e.g. [12, Chapter 2, Theorem 1A (appendix), p.164].

It is customary to define the transition matrix (of $A(.)$) as the map $(t_1, t_2) \mapsto \phi(t_2, t_1) \in \mathbb{R}^{d\times d}$ such that, for any $t_0 \in \mathbb{R}$ and $z_0 \in \mathbb{R}^d$, $t \mapsto z(t) = \phi(t, t_0)z_0$ is the solution of $\dot{z} = A(t)z$, $z(t_0) = z_0$; it satisfies

$$\frac{\partial\phi}{\partial t_2}(t_2, t_1) = A(t_2)\,\phi(t_2, t_1)\,, \quad \frac{\partial\phi}{\partial t_1}(t_2, t_1) = -\phi(t_2, t_1)\,A(t_1)\,, \quad \phi(t, t) = I\,. \tag{53}$$

It is well known that the change of variables $\zeta = \phi(t_0, t)z$ (for any t_0) "kills" the term Az in (51), yielding another formulation of that system:

$$\begin{aligned} \dot{\zeta}(t) &= \phi(t_0, t)B(t)v(t)\,, \\ z(t) &= \phi(t, t_0)\zeta(t)\,, \end{aligned} \tag{54}$$

usually used to derive the general formula for the unique $z(.)$, solution of (51) associated to a prescribed control $v(.)$ and satisfying $z(t_0) = z_0$:

$$z(t) = \phi(t, t_0)\,z_0 + \int_{t_0}^{t} \phi(t, \tau)B(\tau)v(\tau)\mathrm{d}\tau\,. \tag{55}$$

The "unconstrained" case where $V = \mathbb{R}^m$ is fully linear; either the reachable set from any point in any time is the whole state space and controllability holds, or they are all proper affine subspaces, and controllability obviously fails. Criteria are recalled for instance in [11, Section 9.2], after results from [19] or [16]. The classical criterion from [19] states that, for any z_0 in $\mathbb{R}^d$ and t_0, t_f in $\mathbb{R}$, the set $\mathsf{A}^{\mathbb{R}^m}_{t_0,t}(z_0)$ is all $\mathbb{R}^d$ if and only if "the rows of $s \mapsto \phi(t_0,s)B(s)$ are linearly independent on the interval $[t_0,t_f]$", meaning, for $p \in (\mathbb{R}^d)^*$ (a line vector),

$$\Big(p\,\phi(t_0,s)\,B(s) = 0 \text{ for all } s \text{ in } [t_0,t_f]\Big) \quad \Rightarrow \quad p = 0\,. \tag{56}$$

If $A(.)$ and $B(.)$ are real analytic, (56) is equivalent to the columns of the matrices $\big(\frac{\mathrm{d}}{\mathrm{d}t} - A(t)\big)^j B(t)$, $j \in \mathbb{N}$ having rank d for all t in $[t_0,t_f]$ (without the real analyticity assumption, the rank of these columns may vary); this time-varying extension of the Kalman criterion is given in [16].

The case where the convex hull of V is a proper subset of $\mathbb{R}^m$, but still a neighborhood of the origin, is not very different, locally, from the unconstrained case, as noticed in [15] and other references therein. Indeed, in that case, $\mathsf{A}^V_{t_0,t}(0)$ is a neighborhood of the origin if and only if (56) holds (this is stated in [15] in terms of controllability *to* the origin rather than *from* the origin).

The case where the convex hull of V is not a neighborhood of the origin is the purpose of the present paper. Controllability results[9] for this case can be found in [15] and [12, Section 2.2]; we however state here, in a self-contained manner, and together with short proofs, the precise results used in Section 3, focused on deciding when $\mathsf{A}^V_{t_0,t_f}(0)$, or its image by a projection to a smaller dimension space, is a neighborhood of the origin in the state space, or in the above mentioned projection of state space.

Proposition 21. *Assume that V is a compact subset of $\mathbb{R}^m$ containing zero, and that t_0, t_f are two real numbers. The accessible set $\mathsf{A}^V_{t_0,t_f}(0)$ is a neighborhood of the origin if one of the two following equivalent properties hold.*

(i) For $p \in (\mathbb{R}^d)^*$ (line vector),

$$\Big(p\,\phi(t_0,s)\,B(s)v \geq 0 \;\; \textit{for all } v \in V,\ s \in [t_0,t_f]\Big) \quad \Rightarrow \quad p = 0\,. \tag{57}$$

(ii) The convex hull of $\{\phi(t_0,s)\,B(s)v,\ s \in [t_0,t_f],\ v \in V\}$ is a neighborhood of the origin in $\mathbb{R}^d$.

We also give a result of "controllability with respect to a projection", sometimes called partial controllability or output controllability (see *e.g.* [5]), where we are interested in some projection of the accessible set being a neighborhood of the origin in the projection of the state space, rather than the accessible set being a neighborhood of the origin in the state space.

Proposition 22. *Let n be a positive integer no larger than d, and P be a surjective linear map $\mathbb{R}^d \to \mathbb{R}^n$ (or a $n \times d$ matrix with rank n). Assume that V is a compact subset of $\mathbb{R}^m$ containing zero, and that t_0, t_f are two real numbers. The set $P\,\mathsf{A}^V_{t_0,t_f}(0)$ (image of $\mathsf{A}^V_{t_0,t_f}(0)$ by P) is a neighborhood of the origin in $\mathbb{R}^n$ if and only if one of the two following equivalent properties hold.*

(i) For $p \in (\mathbb{R}^n)^*$ (line vector),

$$\Big(p\,P\,\phi(t_f,s)\,B(s)v \geq 0 \;\; \textit{for all } v \in V,\ s \in [t_0,t_f]\Big) \;\Rightarrow\; p = 0\,. \tag{58}$$

(ii) the convex hull of $\{P\,\phi(t_f,s)\,B(s)v,\ s \in [t_0,t_f],\ v \in V\}$ is a neighborhood of the origin in $\mathbb{R}^n$.

[9] The result in [15], characterizes $\cup_{t\in(-\infty,t_f]}\mathsf{A}^V_{t,t_f}(0)$ rather than $\mathsf{A}^V_{t_0,t_f}(0)$ for a fixed t_0.

Proof of Proposition 21. It is a particular case of Proposition 22: taking $M = \mathcal{M}$, $n = d$, and $P = I$ (identity map) in Proposition 22, the conclusion coincides with the one of Proposition 21, while points (i) and (ii) differ in that $\phi(t_0, s)$ is replaced with $\phi(t_f, s)$, but remain equivalent, by factoring out the linear isomorphism $\phi(t_0, t_f)$; for instance, one gets (57) from (58) upon replacing p with $p\,\phi(t_0, t_f)$. ☐

When $n < d$, the controllability property characterized in Proposition 22 is strictly weaker than the one in Proposition 21, and so are conditions (i) and (ii).

Remark 23. If the conditions of Proposition 21 are satisfied for some t_f, they are also satisfied for larger values of t_f (assuming that $\bar{x}(.)$ is defined on some larger time-interval $[t_0, T]$ and one considers its restriction to $[t_0, t_f]$); indeed the set in (57) obviously increases with t_f. It is not the case in Proposition 22, where $\phi(t_0, t_f)$ cannot be factored out, in general, and the conditions may very well be satisfied for one precise value of t_f but not for larger values of t_f; the set in (58) does not increase with t_f unless further assumptions on P are made. If, for instance $P\phi(t_1, t_2) = P$ for any t_1, t_2 (or equivalently $PA(t) = 0$ for all t), one recovers that property, and indeed formulation is simplified in this case because $P\phi(t_f, s)B(s)v$ can be replaced with $PB(s)v$ everywhere.

Proof of Proposition 22. Properties (i) and (ii) are equivalent by the separating hyperplane theorem; let us prove that (i) is necessary and sufficient. If it does not hold, there exists a nonzero p such that $p\,P\phi(t_f, s)\,B(s)v \geq 0$ for any $v \in V$ and $s \in [t_0, t_f]$, and formula (55), with $z_0 = 0$, obviously implies $p\,Pz(t_f) \geq 0$ for any solution $(z(.), v(.))$ of (51) with $z(t_0) = 0$ and $v(.)$ valued in V, implying that $P\,\mathsf{A}^V_{t_0,t_f}(0)$, contained in the half-space $\{y \in \mathbb{R}^n,\ p\,y \geq 0\}$, is not a neighborhood of 0. To prove the converse, assume that $P\,\mathsf{A}^V_{t_0,t_f}(0)$ is *not* a neighborhood of 0 in $\mathbb{R}^n$. As mentioned at the beginning of this appendix, it is convex according to [12, Chapter 2, Theorem 1A (appendix), p.164], hence there is a nonzero $p \in (\mathbb{R}^n)^* \setminus \{0\}$ such that it is contained in the hyperplane $\{y \in \mathbb{R}^n,\ p\,y \geq 0\}$, *i.e.* any solution $(z(.), v(.))$ of (51) with $z(t_0) = 0$ and $v(.)$ valued in V satisfies $pPz(t_f) \geq 0$. Now, let $v \in V$ and $s \in [t_0, t_f)$, and define, for α in $(-\infty, t_f - s]$, the control $u_\alpha : [t_0, t_f] \to V$ by

$$u_\alpha(t) = v \ \text{ if } s < t < s + \alpha\,, \quad u_\alpha(t) = 0 \ \text{ otherwise},$$

and call $z_\alpha(.)$ the unique one such that $(z_\alpha(.), u_\alpha(.))$ is a solution of (51), with $z_\alpha(0) = 0$; from the definition of p, one has $pPz_\alpha(t_f) \geq 0$ for all α. A simple computation using (55) yields

$$\alpha < 0 \Rightarrow z_\alpha(t_f) = 0\,, \quad 0 \leq \alpha \leq t_f - s \Rightarrow z_\alpha(t_f) = \int_s^{s+\alpha} \phi(t_f, \tau)B(\tau)v\,\mathrm{d}\tau\,.$$

The map $\alpha \mapsto pPz_\alpha(t_f)$ is continuous $(-\infty, t_f - s] \to \mathbb{R}$, identically zero on $(-\infty, 0]$, right-differentiable at $\alpha = 0$ with

$$\left.\frac{\mathrm{d}}{\mathrm{d}\alpha}\right|_{\alpha=0^+} pPz_\alpha(t_f) = pP\phi(t_f, s)B(s)v\,;$$

since $pPz_\alpha(t_f)$ remains non negative, this implies $pP\phi(t_f, s)B(s)v \geq 0$; we have proved this inequality for all v in V and all s in $[t_0, t_f]$ ($s = t_f$ was excluded but it is recovered by continuity), and since p is nonzero, this defeats (57). ☐

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