C. R. Acad. Sci. Paris, Ser. I 348 (2010) 1315–1318

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Differential Geometry

Convexity of injectivity domains on the ellipsoid of revolution: The oblate case

Convexité des domaines d'injectivité sur l'ellipsoïde de révolution : le cas oblat

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ARTICLE INFO

Article history:
Received 23 July 2010
Accepted after revision 27 October 2010
Available online 17 November 2010

Presented by Gilles Lebeau

ABSTRACT

We characterize the convexity properties of the tangent injectivity domain on an the ellipsoid of revolution in the oblate case.

RÉSUMÉ

On caractérise les propriétés de convexité du domaine d'injectivité sur un ellipsoïde de révolution oblat.

1. Introduction

The purpose of the present Note is to study convexity properties of injectivity domains on the oblate ellipsoid of revolution given in $\mathbf{R}^3$ by the Cartesian equation

$$E_\mu:\quad x^2+y^2+\left(\frac{z}{\mu}\right)^2=1,$$

with unit semi-major axis and semi-minor axis of length $\mu\in(0,1]$. To this aim, we use the covering of E_μ minus its poles

$$\mathbf{R}\times(0,\pi)\ni(\theta,\varphi)\mapsto(\sin\varphi\cos\theta,\sin\varphi\sin\theta,\mu\cos\varphi),$$

and consider the metric

$$\mathrm{d}s^2=X\,\mathrm{d}\theta^2+(1-X/\lambda)\,\mathrm{d}\varphi^2,$$

where $X=\sin^2\varphi$ and $\lambda=1/(1-\mu^2)\in(1,\infty]$. It can be put in polar form setting

$$\mathrm{d}\psi=\mathrm{d}\varphi\sqrt{1-X/\lambda},$$

which amounts to introducing the elliptic function of second kind $\psi=E(\varphi,k)$ with modulus $k^2=1/\lambda$. According to standard Riemannian geometry [1], for any geodesic but equator and meridians, ψ and φ are periodic with the same period T. In

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1631-073X/$ – see front matter
doi:10.1016/j.crma.2010.10.036

consequence, a geodesic on the oblate ellipsoid of revolution is either a meridian circle, or the equator, or a curve symmetric with respect to the equator such that the ψ component is periodic. Geodesics are integral curves of the Hamiltonian

$$H = \frac{1}{2}\left(\frac{p_\theta^2}{X} + \frac{p_\varphi^2}{1 - X/\lambda}\right),$$

where $p = (p_\theta, p_\varphi)$ is the adjoint vector. The coordinate θ is cyclic, and $p_\theta =$ constant is the Clairaut relation. The Hamiltonian flow allows to define the exponential mapping on $\mathbf{R} \times H(x_0, .)^{-1}(\{1/2\})$ by

$$\exp_{x_0}(t, p_0) = x(t, x_0, p_0).$$

As a subset of the cotangent bundle, the injectivity domain is defined as

$$I(x_0) = \{tp_0 \mid t \in [0, t_{\mathrm{cut}}(x_0, p_0)],\ H(x_0, p_0) = 1/2\},$$

where the cut time $t_{\mathrm{cut}}(x_0, p_0)$ is the supremum of times $t > 0$ such that the curve $\exp_{x_0}(\cdot, p_0)$ is minimizing from $x_0 = \exp_{x_0}(0, p_0)$ to $\exp_{x_0}(t, p_0)$. Since E_μ is symmetric with respect to the equator with Gaussian curvature nondecreasing from the North pole to the equator, the cut time may be shown [4,10] to satisfy (excluding the equator)

$$t_{\mathrm{cut}}(x_0, p_0) = t_{\mathrm{cut}}(p_\theta) = \frac{T(p_\theta)}{2},$$

where $T(p_\theta)$ denotes the period of the φ variable.

Convexity properties are identical whether the domain is expressed on the tangent or on the cotangent bundle since the corresponding change of coordinates is the linear Legendre transform. The domain is convex if and only if its boundary is a convex curve, that is a curve with constant sign curvature. The aim of next section is to sketch a proof of the following:

Theorem 1.1. *The injectivity domain on an oblate ellipsoid of revolution is convex for any point if and only if the ratio between the minor and the major axes is greater or equal to* $1/\sqrt{3}$.

It is worth noticing that the convexity issue plays a crucial role in the regularity theory of optimal transport maps with quadratic cost on Riemannian manifolds (see the monograph [12] for more on optimal transportation). In [9], the convexity of all injectivity domains together with the Ma–Trudinger–Wang condition, is shown to be necessary and sufficient for a compact Riemannian surface to satisfy the so-called *transport continuity property* (Theorem 1.3, in [9]). Therefore, Theorem 1.1 shows that any oblate ellipsoid of revolution with $\mu < 1/\sqrt{3}$ cannot satisfy this property. In [7], this was proven for $\mu \leqslant 0.29 < 1/\sqrt{3}$ (Corollary 5.1, in [7]). According to [8, Theorem 1.1], convexity of all injectivity domains holds for any small enough $\mathcal{C}^4$ perturbation of the round metric on the sphere. As a consequence, the injectivity domains of any oblate ellipsoid of revolution close enough to the round sphere ($\mu = 1$) are always convex. Furthermore, it may be shown that convexity of all injectivity domains is lost in the singular Riemannian case $\mu = 0$ (see the argument below) and a *fortiori* on oblate ellipsoids with small enough semi-minor axis. Theorem 1.1 confirms these facts and shows indeed that the only issue for convexity is the value of μ with respect to $1/\sqrt{3}$.

Besides its own geometric interest, the oblate ellipsoid case is also related to the optimal control of two bodies in space mechanics. It is shown in [2, § 3] that this control problem leads to study on the two-sphere a one-parameter family of metrics[1] which are conformal to the canonical one on an oblate ellipsoid of revolution. This allows to interpretate the parameter as the ratio μ between the minor and major axes of the conformal ellipsoid.[2]

2. Sketch of the proof

Given λ and φ_0 (θ_0 can be set to zero thanks to the symmetry of revolution), the level set $H = 1/2$ is parameterized according to

$$p_\theta = \cos\alpha\sqrt{X_0}, \qquad p_\varphi = \sin\alpha\sqrt{1 - X_0/\lambda}, \quad \alpha \in [0, 2\pi],$$

with $X_0 = \sin^2\varphi_0$. Because of symmetries, convexity has only to be checked on a quarter of the curve, $\alpha \in [0, \pi/2]$. The boundary of the injectivity domain on the cotangent space is

$$\alpha \mapsto \frac{T(p_\theta)}{2}(p_\theta, p_\varphi),$$

so the curvature condition is expressed as a sign condition on the quantity

[1] Also independently introduced in [5].

[2] Surprisingly, the Gauss curvature of these metrics is nondecreasing from the North the pole to the equator (which ensures a simple structure of cut loci [10, Main Theorem]) if and only if μ is again greater or equal to $1/\sqrt{3}$.

$$T\big(T + p_\theta T'\big) + \big(X_0 - p_\theta^2\big)\big(2T'^2 - TT''\big), \quad p_\theta \in [0, \sqrt{X_0}\,],$$

where $' = \mathrm{d}/\mathrm{d}p_\theta$ and where T implicitly also depends on the parameter λ. The quadrature on φ is parameterized by the algebraic complex curve

$$\left[\frac{\dot{X}(\lambda - X)}{\sqrt{\lambda}}\right]^2 = 4\big(X - p_\theta^2\big)(X-1)(X-\lambda)$$

which is of genus one excluding the following degeneracies. When $X_0 = 1$ ($\varphi_0 = \pi/2$), $p_\theta = 1$ defines the equator. The curve also degenerates to a rational surface for $\lambda = \infty$ ($\mu = 1$—round sphere) or $\lambda = 1$ ($\mu = 0$—flat ellipsoid). In the latter case, as the induced metric on the two-sided disk is flat (see [3, Section 4]), the injectivity domain for μ close to 0 and φ_0 close to $\pi/2$ is by continuity a deformation of the union at the origin of two disjoint disks (eight-shaped domain), hence not convex. Conversely, for μ close to 1, the metric is $\mathcal{C}^4$-close to the round one and convexity must hold for an arbitrary initial condition (see [6, Theorem 1.3] or [8, Theorem 1.1]).

Setting

$$u = X - \frac{p_\theta^2 + 1 + \lambda}{3} \quad \text{and} \quad v = \frac{\dot{X}(\lambda - X)}{\sqrt{\lambda}},$$

we get the Weierstrass parameterization by $v^2 = 4u^3 - g_2 u - g_3$ with invariants rational in the parameters,

$$g_2 = \frac{4}{3}\big[p_\theta^4 - (\lambda+1)p_\theta^2 + \big(\lambda^2 - \lambda + 1\big)\big],$$
$$g_3 = \frac{4}{27}\big[2p_\theta^6 - 3(\lambda+1)p_\theta^4 - 3\big(\lambda^2 - 4\lambda + 1\big)p_\theta^2 + \big(2\lambda^3 - 3\lambda^2 - 3\lambda + 2\big)\big].$$

Since X is in $[p_\theta^2, 1]$, u belongs to $[e_2, e_3]$ where the three roots of the cubic are

$$e_1 = \frac{2\lambda - p_\theta^2 - 1}{3}, \qquad e_2 = \frac{2p_\theta^2 - 1 - \lambda}{3}, \qquad e_3 = \frac{2 - p_\theta^2 - \lambda}{3}.$$

The parameterization uses the bounded component of the real cubic, that is $z \in \omega' + \mathbf{R}$ where $\omega\mathbf{Z} + \omega'\mathbf{Z}$ is the real rectangular lattice of periods of $u(z) = \wp(z)$. The time law is

$$\frac{\mathrm{d}t}{\mathrm{d}z} = \frac{\lambda - X}{\sqrt{\lambda}}$$

so the period of φ, which is twice the period of X, equals

$$T = \frac{4}{\sqrt{\lambda}}(e_1\omega + \eta), \quad \eta = \zeta(\omega).$$

Differentiating the period ω and the quasi-period η with respect to the invariants g_2, g_3 (see [11, pp. 307–308]), the derivatives in p_θ are obtained as linear combinations in ω and η with coefficients in $\mathbf{R}(\lambda, p_\theta)$. One has

$$\partial\omega = -A\omega - B\eta, \qquad \partial\eta = C\omega + A\eta,$$

where

$$\partial = \delta\frac{\mathrm{d}}{\mathrm{d}p_\theta}, \qquad \delta = \frac{3(p_\theta^2 - 1)(p_\theta^2 - \lambda)}{p_\theta},$$
$$A = 2p_\theta^2 - (\lambda+1), \qquad B = 3, \qquad C = \frac{1}{3}\big[p_\theta^4 - (\lambda+1)p_\theta^2 + \big(\lambda^2 - \lambda + 1\big)\big].$$

Define $\tau = 3T\sqrt{\lambda}/4$. The derivatives up to second order of τ are

$$\tau = -\big(p_\theta^2 - 2\lambda + 1\big)\omega + 3\eta, \qquad \tau' = \frac{p_\theta}{p_\theta^2 - 1}\big[-\big(p_\theta^2 + \lambda - 2\big)\omega + 3\eta\big],$$
$$\tau'' = \frac{[(2\lambda-1)p_\theta^4 - (\lambda^2+1)p_\theta^2 - \lambda(\lambda-2)]\omega + 3[(\lambda-2)p_\theta^2 + \lambda]\eta}{(p_\theta^2 - 1)^2(p_\theta^2 - \lambda)}.$$

We only provide a sketch of proof that involves the two following lemmas:

Lemma 2.1. *For any $\lambda > 1$ and $X_0 \in [0, 1)$, the (normalized) curvature*

$$\kappa(p_\theta, X_0, \lambda) = \tau\big(\tau + p_\theta\tau'\big) + \big(X_0 - p_\theta^2\big)\big(2\tau'^2 - \tau\tau''\big)$$

is decreasing in p_θ on $[0, \sqrt{X_0}\,]$.

The worst case is so obtained for $p_\theta = \sqrt{X_0}$, and the sign of κ_2 defined according to

$$\kappa_2(X_0, \lambda) = \tau + \tau' \sqrt{X_0}$$

has to be checked for $X_0 \in [0, 1]$.

Lemma 2.2. *For any $\lambda > 1$, κ_2 is decreasing in X_0 on $[0, 1)$.*

The function κ_2 degenerates as $X_0 \to 1$ (equator). Now,

$$\kappa_2 = \frac{\omega}{1 - X_0}\left[(\lambda - 1 - 3\chi) + (\lambda - 2 + 6\chi)(1 - X_0) + 2(1 - X_0)^2\right], \quad \chi = \frac{\eta}{\omega}\cdot$$

The degeneracy of χ is known [11, p. 314], and the previous differentiation rules imply

$$\partial\chi = C + 2A\chi + B\chi^2$$

which allows to obtain an asymptotic of first order of χ when $X_0 \to 1$. Finally,

$$\kappa_2 = \frac{3\pi}{2\sqrt{\lambda - 1}}\left(\lambda - \frac{3}{2}\right) + o(1), \quad X_0 \to 1,$$

hence the zero for $\lambda = 3/2$, that is for $\mu = 1/\sqrt{3}$. As a result, domains of injectivity of the oblate ellipsoid are convex (for any initial condition) if and only if the semi-minor axis is not less than $1/\sqrt{3}$. Below this limit, there are always initial conditions such that convexity is lost.

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