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SECOND ORDER OPTIMALITY CONDITIONS WITH APPLICATIONS

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Abstract. An algorithm to compute the first conjugate point along a smooth extremal curve is presented. Under generic assumptions, the trajectory ceases to be locally optimal at such a point. An implementation of this algorithm, called `cotcot`, is available online and based on recent developments in geometric optimal control. It is applied to analyze the averaged optimal transfer of a satellite between elliptic orbits.

1. **Introduction.** Given a smooth optimal control problem, the well known Pontryagin Maximum Principle [10] provides mainly first-order necessary conditions for optimality which allow to compute optimal trajectories. Numerically, their computation is based on the shooting method (see, *e.g.*, [12]). For a trajectory solution of the maximum principle, second-order conditions based on the notion of conjugate point characterize its local optimality (see [6] and references therein for a recent survey on this theory which has been developed by many authors). Basically, a solution of the maximum principle is locally optimal up to its first conjugate point, and loses its optimality beyond this point.

In this article we first recall shortly in §2 the main definitions attached to the notion of conjugate point, and provide some simple algorithms so as to compute them. These algorithms have been gathered in [6] in a general framework, and have been implemented in the `cotcot` code.[1]

The second section is devoted to the theoretical and numerical investigation of the averaged coplanar energy minimization orbit transfer between Keplerian orbits. The standard energy minimization problem for the coplanar orbit transfer is an optimal control problem in dimension four, and it has been proved in [4] that averaging

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[1]Available at `www.n7.fr/apo/cotcot`.

with respect to the longitude coordinate actually reduces the problem to a three-dimensional Riemannian problem which shares very nice tractable geometric and integrability properties. Moreover, this averaged problem happens to be a very good approximation of the original one. Its study is therefore of primary importance.

After having provided the main geometric features of the problem, we present numerical simulations, led with the aforementioned code, which allow to compute the whole conjugate locus. Combined with the specific geometric features of the averaged orbit transfer problem, global optimality of the geodesics can be characterized in this case.

2. Geometric foundations of the method.

2.1. Maximum principle. We consider the time optimal control problem with fixed extremities x_0, x_f, for a smooth system written $\dot{x} = F(x,u)$ in local coordinates on the *state-space* X and the *control domain* U that are manifolds of dimension n and m, respectively.

Proposition 1. *If (x,u) is an optimal pair on $[0,t_f]$, then there exists an absolutely continuous adjoint vector function p, valued in the cotangent space of X and such that, with $H = \langle p, F(x,u)\rangle$, almost everywhere on $[0,t_f]$ we have*

$$\dot{x} = \frac{\partial H}{\partial p}(x,p,u), \quad \dot{p} = -\frac{\partial H}{\partial x}(x,p,u), \tag{1}$$

and

$$H(x,p,u) = \max_{v\in U} H(x,p,v). \tag{2}$$

Definition 1. The mapping H from $T^*X \times U$ to $\mathbf{R}$ is called the *pseudo-Hamiltonian.* A triple (x,p,u) solution of (1) and (2) is called an *extremal trajectory.*

2.2. Micro-local resolution. Since U is a manifold, restricting to a chart we may assume that, locally, $U = \mathbf{R}^m$, and the maximization condition leads to $\partial H/\partial u = 0$. Our first assumption is the *strong Legendre condition.*

(A1) The quadratic form $\partial^2 H/\partial u^2$ is negative definite along the reference extremal.

Using the implicit function theorem, the extremal control is then locally defined as a smooth function of $z = (x,p)$. Plugging this function into the pseudo-Hamiltonian defines a smooth *true Hamiltonian*, still denoted H, and the extremal is solution of

$$\dot{z} = \overrightarrow{H}(z)$$

with appropriate initial conditions $z_0 = (x_0, p_0)$.

2.3. The concept of conjugate point.

Definition 2. Let $z = (x,p)$ be the reference extremal defined on $[0,t_f]$. The variational equation

$$\delta\dot{z} = d\overrightarrow{H}(z(t))\delta z$$

is called the *Jacobi equation.* A *Jacobi field* is a non-trivial solution $\delta z = (\delta x, \delta p)$. It is said to be *vertical* at time t if $\delta x(t) = d\Pi(z(t))\delta z(t) = 0$ where $\Pi : T^*X \to X$ is the canonical projection.

The following standard geometric result is crucial [5].

Proposition 2. *Let L_0 be the fiber $T^*_{x_0}X$, and let $L_t = \exp t\overrightarrow{H}(L_0)$ be its image by the one-parameter subgroup generated by H. Then L_t is a Lagrangian manifold whose tangent space at $z(t)$ is spanned by Jacobi fields vertical at $t = 0$. Moreover, the rank of the restriction to L_t of Π is at most $n - 1$.*

We introduce the notion of endpoint mapping which will be used to formulate the relevant generic assumptions.

Proposition 3 (see, *e.g.*, [7])**.** *Let $x(t, x_0, u)$ be the response to the control function u starting from x_0 at $t = 0$, and let $E_{x_0,t} : u \mapsto x(t, x_0, u)$ be the* endpoint mapping *for a fixed positive time t. Then, an extremal control u is a singularity of the endpoint mapping when the set of admissible controls is endowed with the L^∞-norm topology. Moreover, the adjoint vector $p(t)$ is orthogonal to the image of $dE_{x_0,t}(u)$.*

In order to derive second order optimality conditions, we make additional generic assumptions.

- **(A2)** On each subinterval $[t_0, t_1]$, $0 < t_0 < t_1 \leq t_f$, the singularity of $E_{x(t_0),t_1-t_0}$ is of codimension one.
- **(A3)** We are in the *normal case*, $H \neq 0$.

As a result, on each subinterval $[t_0, t_1]$, the extremal trajectory admits a unique extremal lift (x, p, u) on the cotangent bundle.

Definition 3. We define the *exponential mapping*

$$\exp_{x_0,t}(p_0) = \Pi(z(t, x_0, p_0))$$

as the projection on X of the integral curve of H with initial condition $z_0 = (x_0, p_0)$, p_0 being restricted to an $(n - 1)$-dimensional submanifold of $T^*_{x_0}X$, depending on the normalization of H.

Definition 4. Let $z = (x, p)$ be the reference extremal. Under our assumptions, the positive time t_c is called *conjugate* if the mapping $\exp_{x_0,t_c}$ is not an immersion at p_0. The associated point $x(t_c)$ is said to be *conjugate* to x_0. We denote by t_{1c} the first conjugate time, and $C(x_0)$ the *conjugate locus* formed by the set of first conjugate points.

The analysis of optimality is done as follows [11].

Proposition 4. *Under our assumptions, the reference extremal is locally time optimal for the L^∞-norm topology on the control set, up to the first conjugate time.*

2.4. **Testing conjugacy.** We provide two equivalent tests.

Test 1. Consider the vector space of dimension $n - 1$ generated by the Jacobi fields $\delta z_i = (\delta x_i, \delta p_i)$, $i = 1, n-1$, vertical at $t = 0$ and such that the $\delta p_i(0)$ are orthogonal to p_0. At a conjugate time t_c, one has

$$\operatorname{rank}\{\delta x_1(t_c), \ldots, \delta x_{n-1}(t_c)\} < n - 1.$$

Test 2. An equivalent test is to augment the family with $\delta z_n = (\dot{x}(t), \dot{p}(t))$ that corresponds to a time variation δt, and the test becomes

$$\delta x_1(t_c) \wedge \cdots \wedge \delta x_{n-1}(t_c) \wedge \dot{x}(t_c) = 0.$$

2.5. **Central field, sufficient conditions and shooting.** We assume that the reference trajectory $t \mapsto x(t)$ is one-to-one, and that there exists no conjugate point on $[0, t_f]$. Then we can imbed locally the reference extremal into a *central field* $\mathscr{F}$ formed by all extremal trajectories starting from x_0. Let $\tilde{H} = p^0 + H$ be the cost extended Hamiltonian where p^0 is normalized to -1 in the normal case. Restricting ourselves to the zero level set, we introduce

$$L = \{\tilde{H} = 0\} \bigcap \bigcup_{t \geq 0} L_t.$$

By construction, $\mathscr{F}$ is the projection of L on the state-space X. The following is clear.

Proposition 5. *The submanifold L is Lagrangian and the projection Π is regular along the reference extremal. There exists a generating map V such that L is locally the graph $\{x,\ p = \partial V(x)/\partial x\}$.*

Using this geometric construction we deduce the standard sufficient condition hereafter [3].

Proposition 6. *Excluding the initial extremity x_0, there exists an open neighbourhood W of the reference trajectory and two smooth mappings $V : W \to \mathbf{R}$, $\hat{u} : W \to U$ such that, for each (x, u) in $W \times U$ the maximization condition*

$$\tilde{H}(x, dV(x), \hat{u}(x)) \geq \tilde{H}(x, dV(x), u)$$

holds, as well as $\tilde{H}(x, dV(x), \hat{u}(x)) = 0$. The reference trajectory is optimal among all trajectories of the system with the same extremities and contained in W.

The shooting mapping of the problem with fixed extremities x_0, x_f is defined in a micro-local neighbourhood of the reference extremal by

$$S(t, p_0) = \exp_{x_0, t}(p_0) - x_f.$$

According to the previous paragraph, our assumptions ensure that it is a smooth function of full rank. This is the crucial condition to compute numerically the trajectory by a shooting method, using for instance a smooth numerical continuation.

The cotcot code. The aim of the method is to provide numerical tools so as to

(i) integrate the smooth Hamiltonian,
(ii) compute Jacobi fields along the extremal,
(iii) detect the resulting conjugate points,
(iv) solve the problem finding a zero of the shooting function.

3. Application to the averaged optimal orbit transfer.

3.1. **Global optimality results in Riemannian geometry.** Consider a Riemannian problem on an n-dimensional manifold state space X, written as

$$\dot{x} = \sum_{i=1}^{n} u_i F_i(x), \quad l = \int_0^{t_f} \left(\sum_{i=1}^{n} u_i^2 \right)^{1/2} dt \to \min$$

where l is the *length* of a curve, and $F_1, \ldots, F_n$ smooth vector fields. Then $g = \sum_{i=1}^{n} u_i^2$ defines a Riemannian metric on X, and $\{F_1, \ldots, F_n\}$ form an orthonormal frame. If we introduce the *energy* $E = \int_0^{t_f} \sum_{i=1}^{n} u_i^2 dt$, the length minimization is well-known [7] to be equivalent to the energy minimization problem (with fixed final time), and to the time minimal control problem if the curves are parameterized by arc-length by prescribing the control on the unit sphere, $\sum_{i=1}^n u_i^2 = 1$.

We can easily compute the extremals for the energy minimization problem. In the normal case ($p^0 = -1/2$, here), the pseudo-Hamiltonian is indeed

$$H = -\frac{1}{2}\sum_{i=1}^n u_i^2 + \sum_{i=1}^n u_i P_i(x,p)$$

where the P_i's are the Hamiltonian lifts $\langle p, F_i(x)\rangle$, $i = 1, n$. The maximization condition leads to $u_i = P_i$ for $i = 1, n$, so that the true Hamiltonian is

$$H(x,p) = \frac{1}{2}\sum_{i=1}^n P_i^2(x,p),$$

which is quadratic in p with full rank.

Definition 5. The *separating line* $L(x_0)$ is the set where two minimizing extremals departing from x_0 intersect. The *cut locus* $\mathrm{Cut}(x_0)$ is the set of points where extremals cease to be globally optimal. We note $i(x_0)$ the distance to the cut locus,

$$i(x_0) = d(x_0, \mathrm{Cut}(x_0)),$$

and $i(X)$ the *injectivity radius*,

$$i(X) = \inf_{x_0 \in X} i(x_0).$$

We summarize now some useful properties of these sets [9].

Proposition 7. *Assume the Riemannian metric complete. Then,*

(i) *A cut point is either on the separating line, or a conjugate point.*
(ii) *If x_1 is a point that realizes the distance $i(x_0)$, then either x_1 is conjugate to x_0, or there are two minimizing geodesics joining x_0 to x_1 that form the two halves of the same closed geodesic.*
(iii) *The distance $i(x_0)$ is the smallest r such that the Riemannian sphere $S(x_0, r)$ is not smooth.*

3.2. **Coplanar orbital transfer and averaging.** We sketch the framework of [4] to which we refer the reader for details. We consider the coplanar orbit transfer represented in *Gauss coordinates* (l, x) where l is the *longitude* (polar angle of the satellite), and where x contains the *orbit elements* [8], *e.g.*, $x = (P, e_x, e_y)$, P being the *semi-latus rectum*, and (e_x, e_y) the *eccentricity vector* whose direction is the semi-major axis and whose length e is the standard eccentricity. The system decomposes into

$$\dot{x} = u_1 F_1(l,x) + u_2 F_2(l,x), \quad \dot{l} = g_0(l,x),$$

with

$$\begin{aligned}
F_1 &= P^{1/2}\left(\sin l \frac{\partial}{\partial e_x} - \cos l \frac{\partial}{\partial e_y}\right)\\
F_2 &= P^{1/2}\left[\frac{2P}{W}\frac{\partial}{\partial P} + \left(\cos l + \frac{e_x + \cos l}{W}\right)\frac{\partial}{\partial e_x} + \left(\sin l + \frac{e_y + \sin l}{W}\right)\frac{\partial}{\partial e_y}\right],
\end{aligned}$$

and $W = 1 + e_x \cos l + e_y \sin l$, $g_0 = W^2/P^{3/2}$. This system of coordinates is well defined on the *elliptic domain* $X = \{P > 0,\ |e| < 1\}$ filled by elliptic trajectories of the controlled Kepler equation. The boundary $|e| = 1$ corresponds to parabolic trajectories whereas $e = 0$ defines circular orbits.

We consider the energy minimization problem with cost $\int_0^{t_f}(u_1^2+u_2^2)dt$. The final longitude is free and can be used to reparameterize the trajectories. The true Hamiltonian in the normal case is obtained as before,

$$H(l,x,p)=\frac{1}{2g_0(l,x)}\sum_{i=1}^{2}P_i(l,x,p)^2,\quad P_i(l,x,p)=\langle p,F_i(l,x)\rangle.$$

We observe that H is 2π-periodic with respect to l, and this leads to introduce the *averaged Hamiltonian.* Such a system is related to an approximation of the true trajectories by the standard theory [1].

Definition 6. The averaged Hamiltonian is

$$\overline{H}(x,p)=\frac{1}{2\pi}\int_0^{2\pi}H(l,x,p)dl.$$

It is evaluated using the residue theorem and we get the result hereafter [4].

Proposition 8. *Let $P=(1-e^2)/n^{2/3}$, $e_x=e\cos\theta$ and $e_y=e\sin\theta$, where n is the* mean movement *and θ the argument of perigee. In the new coordinates $x=(n,e,\theta)$, the averaged Hamiltonian is*

$$\overline{H}=\frac{1}{4n^{5/3}}\left[18n^2p_n^2+5(1-e^2)p_e^2+(5-4e^2)\frac{p_\theta^2}{e^2}\right].$$

Accordingly, integral curves of $\overline{H}$ are extremals of the Riemannian metric in $\mathbf{R}^3$ defined by

$$ds^2=\frac{1}{9n^{1/3}}dn^2+\frac{2n^{5/3}}{5(1-e^2)}de^2+\frac{2n^{5/3}}{(5-4e^2)}e^2d\theta^2,$$

and (n,e,θ) are orthogonal coordinates.

Our problem is then to analyze the corresponding Riemannian problem to derive optimality results for the original one.

3.3. **Clairaut-Liouville metrics.** An important step in the analysis consists in computing normal coordinates.

Proposition 9. *In the elliptic domain, we set*

$$r=\frac{2}{5}n^{5/6},\quad \varphi=\arcsin e.$$

In these coordinates, the metric of the averaged problem is

$$ds^2=dr^2+\frac{5}{2}r^2(G(\varphi)d\theta^2+d\varphi^2) \tag{3}$$

where

$$G(\varphi)=\frac{5\sin^2\varphi}{1+4\cos^2\varphi}.$$

We first observe that metrics of the form (3) reveal two metrics in dimension two, namely

$$g_1=dr^2+\frac{5}{2}r^2d\varphi^2 \text{ and } g_2=G(\varphi)d\theta^2+d\varphi^2.$$

The Hamiltonian associated with the full metric on $\mathbf{R}^3$ is

$$H=\frac{1}{2}p_r^2+\frac{2}{5r^2}H_2,$$

while H_2 is associated with g_2,

$$H_2 = \frac{1}{2}\left(\frac{p_\theta^2}{G(\varphi)} + p_\varphi^2\right).$$

Lemma 1. *Since θ is a cyclic coordinate, p_θ is a first integral. On the level set $\{p_\theta = 0\}$, extremals verify $\theta =$ constant and geodesics of the* polar metric g_1 *are straight lines in coordinates $x = r\sin\psi$, $z = r\cos\psi$, with*

$$\psi = \frac{\varphi}{c}, \quad c = \sqrt{\frac{2}{5}}.$$

To complete the analysis, we introduce the following [2].

Definition 7. A metric of the form $G(\varphi)d\theta^2 + d\varphi^2$, where G is a 2π-periodic function, is called a *Clairaut-Liouville* metric. The level sets of the two angles θ and φ are *meridians* and *parallels*, respectively.

Proposition 10. *The Gaussian curvature of a Clairaut-Liouville metric $G(\varphi)d\theta^2 + d\varphi^2$ is*

$$K = -\frac{1}{\sqrt{G}}\frac{\partial^2\sqrt{G}}{\partial\varphi^2},$$

and the Clairaut relation *holds,*

$$p_\theta = \cos\phi\sqrt{G}$$

where ϕ is the angle of an extremal parameterized by arc-length with a parallel. Morevover, the extremal flow is integrable by quadratures.

We fix the level set to $H = 1/2$. If (r, θ, φ) are the components of an extremal curve, we begin by noting that r^2 is a degree two polynomial of time,

$$r^2(t) = t^2 + 2r(0)p_r(0)t + r^2(0). \tag{4}$$

The remaining equations are then integrated thanks to the reparameterization

$$dT = 2dt/(5r^2) \tag{5}$$

as Clairaut-Liouville extremals.

Regarding global optimality, the crucial point is to relate the cut locus of the three-dimensional metric to the cut locus of the restriction of this metric to the sphere. Let x_1 and x_2 be two extremal curves, both starting from the same initial condition x_0. Since extremals are parameterized by arc-length, finding points in $L(x_0)$ amounts to finding $\bar{t}$ such that $x_1(\bar{t}) = x_2(\bar{t})$. Now, if $r_1(\bar{t}) = r_2(\bar{t})$ then $p_{r,1}(0) = p_{r,2}(0)$ and $r_1(t) = r_2(t)$ for all t by virtue of (4). The analysis is thus reduced to dimension two.

Proposition 11. *The separating line $L(x_0)$ is characterized as follows: There exists two extremals x_1, x_2 with initial condition x_0, and there exists $\overline{T} > 0$ such that*

$$\theta_1(\overline{T}) = \theta_2(\overline{T}), \quad \varphi_1(\overline{T}) = \varphi_2(\overline{T}),$$

and such that, for some finite $\bar{t}$, the following compatibility condition *is fulfilled,*

$$\overline{T} = \int_0^{\bar{t}} \frac{2dt}{5r^2}.$$

3.4. **Application to orbital transfer.** According to the previous reduction, the main point to conclude about global optimality is to evaluate $i(x_0)$, the distance to the cut locus of the Clairaut-Liouville metric $g_2 = G(\varphi)d\theta^2 + d\varphi^2$. A direct computation gives the curvature.

Proposition 12. *The curvature of the Clairaut-Liouville metric g_2 is*

$$K = \frac{5(1-8\cos^2\varphi)}{(1+4\cos^2\varphi)^2} \leq 5.$$

Besides, we can rewrite the metric according to

$$g_2 = E_\mu^{-1}(\varphi)\left(\sin^2\varphi d\theta^2 + E_\mu(\varphi)d\varphi^2\right)$$

with $E_\mu = \mu^2 + (1-\mu^2)\cos^2\varphi$ and $\mu = 1/\sqrt{5}$.

Proposition 13. *The Clairaut-Liouville metric g_2 is conformally equivalent to the flat metric on an ellipsoid of revolution of semi-minor axis $1/\sqrt{5}$.*

The restriction of the flat metric on $\mathbf{R}^3$ to the ellipsoid

$$x = \sin\varphi\cos\theta, \quad y = \sin\varphi\sin\theta, \quad z = \mu\cos\varphi$$

is indeed $ds^2 = \sin^2\varphi d\theta^2 + E_\mu(\varphi)d\varphi^2$.

Using the `cotcot` algorithm, we can compute for both metrics the corresponding conjugate loci. On figure 1, the geodesics associated to orbital transfer are given for $\varphi_0 = \pi/2$. On figure 2, we compare the conjugate loci of the two metrics.

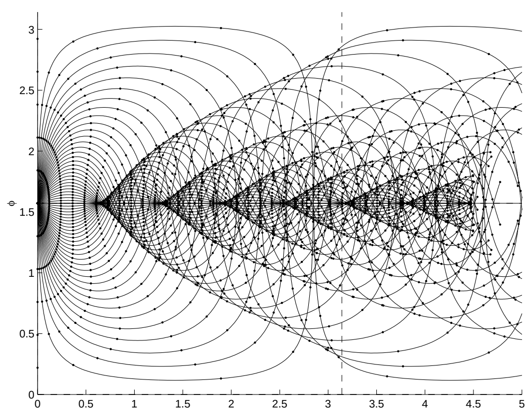

FIGURE 1. Geodesics in orbital transfer for $\varphi_0 = \pi/2$. The conjugate locus appears as the leftmost enveloppe of points.

An important observation is that $\varphi_0 = \pi/2$ is the equator solution where the curvature is constant and maximum: The curvature K is equal to 5, and the first conjugate point has minimum length, $\pi/\sqrt{5}$. On figure 3, we compute the respective spheres for $\varphi_0 = \pi/2$. In orbital transfer, much similarly to the standard situation on the ellipsoid, the sphere is smooth up to length $\pi/\sqrt{5}$ that defines the injectivity radius.

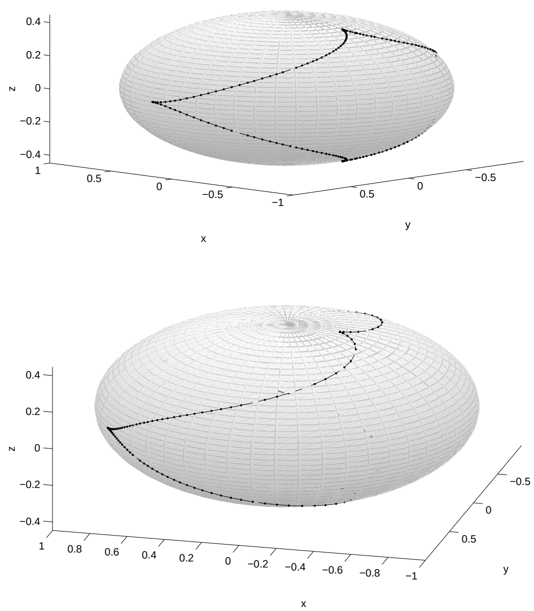

FIGURE 2. Conjugate loci for $\varphi_0 = \pi/2$. Flat case (top) and orbital transfer (bottom). The standard astroid conjugate locus is obtained in the flat case.

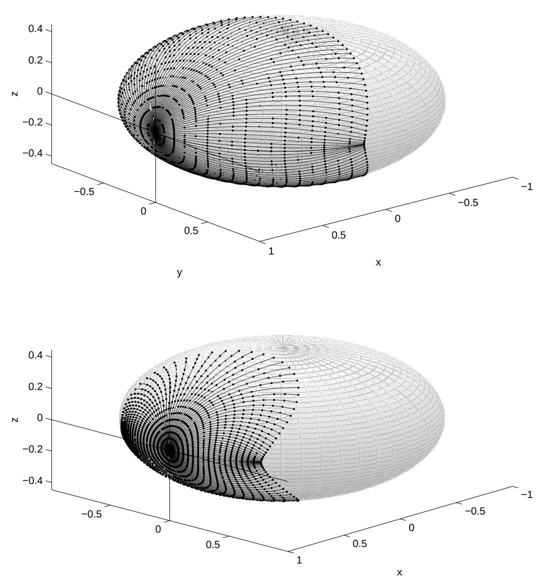

FIGURE 3. Spheres and injectivity radius for $\varphi_0 = \pi/2$. Flat case (top) and orbital transfer (bottom).

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