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DISCRETE AND DIFFERENTIAL HOMOTOPY IN CIRCULAR RESTRICTED THREE-BODY CONTROL

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ABSTRACT. The planar circular restricted three-body problem is considered. The control enters linearly in the equation of motion to model the thrust of the third body. The minimum time optimal control problem has two scalar parameters: The ratio of the primaries masses which embeds the two-body problem into the three-body one, and the upper bound on the control norm. Regular extremals of the maximum principle are computed by shooting thanks to continuations with respect to both parameters. Discrete and differential homotopy are compared in connection with second order sufficient conditions in optimal control. Homotopy with respect to control bound gives evidence of various topological structures of extremals.

Introduction. An important simplification of the general three-body problem in celestial mechanics is the planar circular restricted problem: Two primaries (*e.g.*, Earth and Moon) rotate around the center of mass, while a third body evolves in the plane of their Keplerian motion under their attraction, but not influencing them. In a rotating frame centered on the barycenter of the primaries, the equations of the controlled motion are

$$\ddot{q} + 2\overline{\partial}V_{\mu}(q) + 2i\dot{q} = \varepsilon u, \quad |u| = \sqrt{u_1^2 + u_2^2} \le 1,$$

where q and $u \in \mathbf{C} \simeq \mathbf{R}^2$ are respectively the position and the control vector. The potential is

$$V_{\mu}(q) = -\frac{|q|^2}{2} - \frac{1-\mu}{|q+\mu|} - \frac{\mu}{|q-1+\mu|} \cdot$$

The system has two parameters, the ratio μ of the primaries masses, and the bound ε on the acceleration available to control the third body (a spacecraft). The system has two limit cases: (i) $\mu = 0$, which defines a two-body problem (the second

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primary having no mass); (ii) $\mu = 1/2$ for primaries with equal masses defining a circular choreography.

In cartesian coordinates (q, \dot{q}) , the system is defined on the tangent bundle TQ_{μ} where Q_{μ} is the complex plane with two punctures at the primaries, $-\mu$ and $1-\mu$, (singularities of the potential). The set Q_{μ} has the topology of the eight figure, with a non-trivial fundamental group ($\mathbf{Z} * \mathbf{Z}$). One can alternatively use Jacobi energy first integral,

$$J_{\mu}(q, \dot{q}) = \frac{|\dot{q}|^2}{2} + V_{\mu}(q)$$

to write the uncontrolled system in Hamiltonian form. Setting $p = \dot{q} + iq$,

$$J_{\mu}(q,p) = \frac{|p|^2}{2} + p \wedge q - \frac{1-\mu}{|q+\mu|} - \frac{\mu}{|q-1+\mu|}$$

and the controlled dynamics is

$$\dot{q} = \frac{\partial J_{\mu}}{\partial p}, \quad \dot{p} = -\frac{\partial J_{\mu}}{\partial q} + \varepsilon u, \quad |u| \le 1.$$

In control form,

$$\dot{x} = F_0(x) + \varepsilon(u_1 F_1(x) + u_2 F_2(x)), \quad |u| \le 1.$$

with $x \in X_{\mu}$, and $X_{\mu} = TQ_{\mu}$ or $X_{\mu} = T^*Q_{\mu}$ (cotangent bundle) depending on the choice of coordinates. In (q, \dot{q}) variables,

$$F_{0,\mu}(x) = \dot{q}\frac{\partial}{\partial q} - (2\overline{\partial}V_{\mu}(q) + 2i\dot{q})\frac{\partial}{\partial \dot{q}}, \quad F_1(x) = \frac{\partial}{\partial \dot{q}_1}, \quad F_2(x) = \frac{\partial}{\partial \dot{q}_2}$$

We consider minimum time control of the system. (See [5] for the case of energy). According to Pontryagin maximum principle, optimal trajectories must be projections on X_{μ} of *extremals*, that is of integral curves of the following Hamiltonian:¹

$$H = p^{0} + H_{0,\mu} + \varepsilon (u_{1}H_{1} + u_{2}H_{2}), \quad H_{i} = \langle p, F_{i}(x) \rangle, \quad i = 0, 1, 2.$$

The Hamiltonian and the lifts H_i are defined on T^*X_{μ} , and there are two possibilities because of homogeneity in (p^0, p) $(p^0$ is a nonpositive scalar): (i) $p^0 \neq 0$ (normal case), (ii) $p^0 = 0$ (abnormal case). Restricting to normal extremals, we set $p^0 = -1$. Moreover, the Hamiltonian has to be maximized almost everywhere with respect to the control, so $u = (H_1, H_2)/|(H_1, H_2)|$ and

$$H = -1 + H_{0,\mu} + \varepsilon \sqrt{H_1^2 + H_2^2} \tag{1}$$

outside the codimension two switching surface $\Sigma = \{H_1 = H_2 = 0\}$. (See [8] for a detailed analysis).

The flow of the smooth Hamiltonian (1) is studied using homotopies on the two parameters of the problem. First, "à la Poincaré", with a continuation on parameter μ . This amounts to connecting (controlled) two- and three-body problems, the former being well understood [3]. Discrete and differential homotopies on μ are compared in section 1. A neat framework for the latter is provided by the theory of second order conditions in optimal control. In section 2, homotopy on the control bound ε is considered. Using both discrete and differential homotopies, insight into the corresponding paths is obtained despite difficulties due in particular to the topology of Q_{μ} as discussed in the last section.

¹From now on, p denotes the dual to x, not to q anymore: $(x, p) \in T^*X_{\mu}$.

1. Homotopy on μ . Continuation on μ allows to embed the two-body controlled problem into the three-body one. The simplest approach is then to devise heuristically a sequence $(\mu_k)_k$ from $\mu = 0$ to the targeted value μ , for instance $\mu = 1/2$. At step k, the solution is used to initialize the (k + 1)-th computation. Good initializations are indeed crucial to ensure convergence of the so-called *shooting method* based on Pontryagin maximum principle.

Since the final time to be minimized is unknown, extremals are to be sought on the zero level set of the Hamiltonian. One defines the *exponential mapping*

$$\exp_{x_0} : (t_f, p_0) \mapsto x(t_f, x_0, p_0)$$

on a neighbourhood of the origin in $\mathbf{R} \times H(x_0, .)^{-1}(\{0\})$, where $x(t_f, x_0, p_0)$ is the projection on the *x*-space of the extremal departing from (x_0, p_0) . Expliciting the dependence with the homotopic parameter $\lambda = \mu$ (ε is fixed), the *shooting function* is

$$S(\xi,\lambda) = \exp_{x_0(\lambda)}(\xi,\lambda) - x_f(\lambda), \quad \xi = (t_f, p_0),$$

where x_f is the target (a choice of coordinates in which to express the terminal conditions is assumed). Both the initial and final points may vary with μ , here, hence the dependence of x_0 and x_f in the parameter.² Numerical results of discrete homotopy to compute time minimal trajectories towards the L_2 Lagrange libration point (equilibrium point of the uncontrolled system whose position depend on μ , see [12]) are displayed in Fig. 1. A remarkable feature is that, starting from $\mu = 0$, one discrete step suffices to reach $\mu \simeq 1.21e - 2$ corresponding to the Earth-Moon system. Local optimality of the computed extremals is checked using the theory of conjugate points that we now briefly sketch [1].



FIGURE 1. Homotopy on μ , $\varepsilon = 2.440e - 1$, minimum time transfer towards L_2 Lagrange point. Left, $\mu = 0.1$. The first phase around the first primary is essentially a two-body control phase. This is typically the case for the Earth-Moon system as $\mu \simeq 1.21e - 2$ is even smaller. Right, $\mu = 0.5$. The perturbation due to the second primary now with equal mass is clearly observed as the eccentricity of intermediate orbits is increased.

 $^{^2\}mathrm{The}$ target alone will depend on the homotopy parameter, here.

An extremal z = (x, p) is *regular* if it exists $\alpha > 0$ such that the strong Legendre condition holds along z (below m is the dimension of the control),

$$\frac{\partial^2 H}{\partial u^2}(x(t), p(t), u(t))(v, v) \le -\alpha |v|^2, \quad v \in \mathbf{R}^m, \quad t \in [0, t_f].$$

Here, as u belongs to S^1 , one can set $u = (\cos \theta, \sin \theta)$ ($\theta \in \mathbf{R}$), and extremals are regular outside the switching surface as

$$\frac{\partial^2 H}{\partial \theta^2} = -\varepsilon \sqrt{H_1^2 + H_2^2} \,.$$

The variational equation on $[0, t_f]$, $\dot{\delta z}(t) = \vec{H}'(z(t))\delta z(t)$ is called the Jacobi equation along z. An instant $t_c \in (0, t_f]$ is a conjugate time (and $x(t_c)$ is a conjugate point) if there is a Jacobi field, $\delta z = (\delta x, \delta p)$, non-trivial (δx not $\equiv 0$) and vertical at 0 and t_c ,

$$\delta x(0) = \delta x(t_c) = 0.$$

Theorem 1.1. [1] A normal regular extremal without conjugate point is \mathcal{C}^0 -locally optimal among trajectories with same endpoints.

By restricting if necessary the open subset Ω of definition of the shooting function, we may assume that S is regular. Then, 0 is a regular value and each connected component of $S^{-1}(\{0\})$ (there are finitely many if Ω is bounded) is a one-dimensional embedded submanifold diffeomorphic either to \mathbf{S}^1 or to \mathbf{R} . As each $c = (\xi, \lambda) \in \Omega$ is a regular point of S, it is possible to define $\overrightarrow{T}(c)$ as the unit vector in the kernel of S'(c). Such a vector is unique provided a choice of orientation is made according to det(${}^tS'(c), \overrightarrow{T}(c)$) > 0 (see [2]). Starting from $c_0 \in S^{-1}(\{0\})$, the corresponding connected component or *branch* is parameterized by the differential equation with respect to curvilinear abscissa

$$\frac{\mathrm{d}c}{\mathrm{d}s}(s) = \overrightarrow{T}(c(s)), \quad c(0) = c_0.$$
(2)

At a point $c(s) = (t_f(s), p_0(s), \lambda(s))$, second order sufficient conditions are checked using a simple numerical rank evaluation [7]. Indeed, at time $t \in (0, t_f(s)]$, Jacobi fields span the image of the differential with respect to (t, p_0) of

$$(t, p_0, \lambda) \mapsto x(t, x_0, p_0, \lambda).$$

So verifying that this mapping has a full rank derivative at all $(t, p_0(s))$ for $t \in (0, t_f(s)]$ ensures that we have a \mathscr{C}^0 -local minimizer at $t_f(s)$. This eventually implies that $\partial S/\partial \xi(c(s))$ has full rank, so the path can locally be parameterized not only by s but also by λ (no *turning point*). The path computed thanks to [7] which, in contrast to standard differential homotopy codes, integrates without correction step equation (2), is portrayed in Fig. 2. The strength of the differential approach, when applicable, is that steps are automatically computed and tuned.

2. Homotopy on ε . Having obtained three-body minimum time trajectories using an homotopy on μ , our second goal is to iterate these computations for smaller values of the control bound,³ ε . We are faced with several issues. Paths may have turning points associated with rank drops in the partial derivative of the shooting function that prevent the verification of second order sufficient conditions (see the end of the section for an adaptation of such conditions to the case of terminal

 $^{^{3}{\}rm Low}$ thrust transfers are an important class of applications in space mechanics. The SMART-1 mission is a recent example of such missions.



FIGURE 2. Homotopy on μ , $\varepsilon = 2.440e - 1$, minimum time transfer towards L_2 Lagrange point. The value function $\mu \mapsto t_f(\mu)$ is portrayed.

submanifold—a circular orbit around the Moon is targeted, here). See for instance Fig. 3-4. Moreover, to ensure global optimality, for a given ε one has to compare costs on the different connected components of the set of zeros, and it turns that one has to jump from one component to another. While a given branch is followed using differential homotopy (integration of 2), switching branches is obtained by discrete homotopy. Combining both approaches allow to obtain extremals whose local optimality is checked by second order conditions and whose global optimality is tested by comparing various branches.



FIGURE 3. Homotopy on ε , Earth-Moon case ($\mu \simeq 1.21e - 2$), minimum time transfer towards a circular orbit around the second primary. On both subgraphs, ε is expressed in Newtons. In the Earth-Moon system, a 1 Newton thrust on a 1500 Kilogram spacecraft (typical mass for medium or low-thrust transfers) is equivalent to $\varepsilon \simeq 2.440e - 1$. The same reference to Newtons is made in subsequent figures. On the left, the abscissa is the curvilinear abscissa on the path and two turning points are observed (see also Fig. 4). The corresponding cusp points on $\varepsilon \to t_f(\varepsilon)$ are observed on the right. Optimality is necessarily lost past the self-intersection point near $\varepsilon \simeq 9.675e - 1$ (see also Fig. 9).

Before proposing a preliminary discussion of the topology of solutions observed on different branches (see Fig. 5), we conclude the section by recalling that, in the



FIGURE 4. Homotopy on ε , Earth-Moon case ($\mu \simeq 1.21e-2$), minimum time transfer towards a circular orbit around the second primary. The four components of the adjoint state p_0 are portrayed. The same two turning points as in Fig. 3 are observed. A refined computation confirm that there is no self-intersection of the curves (no bifurcation).

case of a terminal condition defining a submanifold (here a circular orbit instead of a single point such as a Lagrangian point as in section 2), the conjugate point notion can be adapted as follows. When the target is a proper submanifold X_f , a time $t_c \in (0, t_f]$ is said to be *focal* (see [11] for a discussion in the Riemannian case, and [4] in optimal control) if there is a non-trivial Jacobi field δz along z which is vertical at t = 0 and such that

$$\delta z(t_c) \in T_{z(t_c)} X_f^\perp$$

where X_f^{\perp} is the Lagrangian submanifold

$$X_f^{\perp} = \{ z = (x, p) \in T^*X \mid x \in X_f \text{ and } p \perp T_x X_f \}.$$

We are thus again able to check local optimality of transfers from circular to circular orbits, such as the one in Fig. 6.

3. Homological classification. Homotopy on ε in the previous section suggests that to different connected components of the path are associated solutions of different homological nature: Either winding around the second primary changes from positive to negative, or the sign remain constant but the winding number changes. A coarse classification can be made using homology of (closed) curves in Q_{μ} . (The

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FIGURE 5. Homotopy on ε , Earth-Moon case ($\mu \simeq 1.21e-2$), minimum time transfer towards a circular orbit around the second primary. Top pictures are transfers for $\varepsilon = 2.440e - 1$ (left) and $\varepsilon = 2.342e - 1$ (right), respectively. Bottom pictures are transfers for $\varepsilon = 2.294e - 1$ (left) and $\varepsilon = 2.196e - 1$ (right), respectively. While the two previous trajectories wind up positively around the two primaries, both transfers wind up negatively around the second primary, illustrating a first difference of homological nature. Moreover, the last transfer differs from the three others in having one more revolution around the first primary.

classification is coarse as it takes into account neither the \dot{q} coordinate, nor the adjoint p).

First restricting to extremals with boundary conditions on $q_2 = 0$ (the line defined by the two punctures in Q_{μ}), one can associate to any such extremal a closed curve by patching to it the curve symmetric with respect to $q_2 = 0$ (and orientation reversed), see Fig. 7. The homology of this curve turns to be sufficient to classify the different types of solutions observed on disjoint components of the homotopy path. Note nevertheless that in Fig. 5, the first two curves (upper part) belong to the same branch, have the same winding number around the first singularity but not around the second one (though the winding sign is unchanged). For arbitrary boundary conditions, the same symmetric closure has to be performed on the truncation of the q-projection of the extremal from its first intersection with $q_2 = 0$ until the last one. As a case study, the path between $\varepsilon = 2.441e - 1$ and $\varepsilon = 2.196e - 1$ is discussed in Fig. 8. The optimality status of each homology class is analyzed in



FIGURE 6. Earth-Moon minimum time transfer towards a circular orbit around the second primary, fixed frame, $\varepsilon = 2.220e - 1$. Controlled trajectory in blue, free trajectory (null control after the target is reached) green. The capture into a quasi-periodic motion around the second primary is clearly observed.

Fig. 9. It is a matter of future investigation to integrate homological constraints [10] to cut out sub-optimal components from the path.



FIGURE 7. Closure by symmetrization. The extremal (blue) is truncated if necessary so that endpoints lie on $q_2 = 0$. Concatenating the symmetric curve (red) with reversed orientation results in a closed curve whose homotopy class and homology are well defined—(7, -1), here.



FIGURE 8. Homotopy on ε , Earth-Moon case ($\mu \simeq 1.21e-2$), minimum time transfer towards a circular orbit around the second primary. Top pictures are transfers for $\varepsilon = 2.440e - 1$ (left) and $\varepsilon = 2.206e - 1$ (right), respectively. The two extremals belong to the same branch, labeled I (see Fig. 3) and have the same homology (positively winding up around both primaries). Middle pictures are transfers for $\varepsilon = 2.440e - 1$ (left) and $\varepsilon = 2.197e - 1$ (right), respectively. Extremals belong to the same branch, labeled II, and have the same homology (now negatively winding up around the second primary). Bottom pictures are transfers for $\varepsilon =$ 2.440e-1 (left) and $\varepsilon = 2.196e-1$ (right), respectively. Extremals belong to the same branch, labeled III, and have the same homology (negatively winding up around the second primary with one more revolution around the first one that those on branches I and II).



FIGURE 9. Homotopy on ε , Earth-Moon case ($\mu \simeq 1.21e - 2$), minimum time transfer towards a circular orbit around the second primary. Each curve represents the function $\varepsilon \mapsto t_f(\varepsilon)$ evaluated along the three disjoint branches I (blue), II (red) and III (black) of the homotopy (see Fig. 8). Slightly below $\varepsilon \simeq 2.367e - 1$ (that is below 0.97 Newtons on the graph), branch II provides a better criterion than I, but both have already become worse than branch III. The infimum of the three curves provides an upper bound for the value function $\varepsilon \mapsto t_f(\varepsilon)$.

REFERENCES

- [1] A. A. Agrachev, Yu. L. Sachkov, Control theory from the geometric viewpoint, Springer, 2004.
- [2] E. L. Allgower and K. Georg, Introduction to numerical continuation methods, SIAM, 2003.
- [3] B. Bonnard, J.-B. Caillau and E. Trélat, Geometric optimal control of elliptic Keplerian orbits, Discrete Contin. Dyn. Syst. Ser. B 5 (2005), no. 4, 929–956.
- [4] B. Bonnard, G. Launay and M. Pelletier, Classification générique de synthèses temps minimales avec cible de codimension un et applications, Ann. Inst. H. Poincaré Anal. Non Linéaire 14 (1997), no. 1, 55–102.
- [5] B. Bonnard and G. Picot, Energy minimal transfers in the vicinity of the L_1 Lagrangian point, this volume.
- [6] B. Bonnard, N. Shcherbakova and D. Sugny, The smooth continuation method in optimal control with an application to quantum systems, ESAIM Control Optim. and Calc. Var. 17 (2010), no. 1, 267–292.
- [7] J.-B. Caillau, O. Cots and J. Gergaud, Differential continuation for regular optimal control problems, Optim. Methods Softw., to appear, apo.enseeiht.fr/hampath.
- [8] J.-B. Caillau, B. Daoud and J. Gergaud, On some Riemannian aspects of two and three-body controlled problems, in "Recent Advances in Optimization and its Applications in Engineering", 205–224, Springer, 2010.

- [9] J.-B. Caillau, J. Gergaud and J. Noailles, 3D Geosynchronous Transfer of a Satellite: Continuation on the Thrust, J. Optim. Theory Appl. 118 (2003), no. 3, 541–565.
- [10] A. Chenciner, De l'espace des triangles au problème des trois corps, Gaz. Math. 104 (2005), 22–38.
- [11] M. P. do Carmo, Riemannian geometry, Birkhäuser, 1992.
- [12] V. Szebehely, Theory of orbits: The restricted problem of three bodies, Academic Press, 1967.

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