1

Optimisation of functional determinants on the circle

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Summary. The functional determinant of elliptic differential operators on the circle was introduced in [3]. In the present paper, optimisation of this determinant over essentially bounded functions is studied as an optimal control problem on the special linear group of real matrices. In the one dimensional case, existence and uniqueness of maximisers and minimisers is proved.

1.1 Statement of the problem

Following [3] we consider the determinant of a differential operator

$$A := \sum_{k=0}^{p} A_k D^k$$

defined on $\mathbf{R}^N$-valued functions, N a positive integer, where $D = -i\mathrm{d}/\mathrm{d}x$ is the complex valued derivation operator for such functions ($i^2 = -1$) and where the $A_k : \mathbf{S}^1 \to \mathrm{M}(N, \mathbf{R})$, $0 \le k \le p$, are matrix-valued (square matrices of order N) functions defined on the circle.[5] We are interested in addressing optimisation issues for such determinants under suitable restrictions on

[5] The fundamental reference for spectral problems on the circle $\mathbf{S}^1$ (geometrisation of the periodic boundary conditions) is [3], more general than [4]. The latter

the potentials involved. For the rest of the paper, we identify $\mathbf{S}^1$ with $\mathbf{R}/\mathbf{Z}$ and functions on $\mathbf{S}^1$ with one-periodic functions. For $Q \in \mathrm{M}(N, \mathbf{R})$ we use the Frobenius norm $\|Q\| = \mathrm{tr}(Q^T Q)^{1/2}$ and recall it derives from the inner product on $\mathrm{M}(N, \mathbf{R})$ given by

$$\langle Q_1, Q_2 \rangle = \mathrm{tr}(Q_1^T Q_2), \quad Q_1, Q_2 \text{ in } \mathrm{M}(N, \mathbf{R}). \tag{1.1}$$

We will assume that

$$A = -\operatorname{Id}_N \frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x), \tag{1.2}$$

i.e., the maximal order of differentiation p is equal to two, and the operator is in normal form with $A_2 = \mathrm{Id}_N$ (the identity matrix of order N), $A_1 = 0$ and $A_0 = V$ a Hill potential. Ray and Singer [6] define the *functional determinant* of such an operator as

$$\det A := e^{-\zeta_A'(0)} \tag{1.3}$$

where

$$\zeta_A(s) := \sum_{\lambda_j > 0} \frac{1}{\lambda_j^s},$$

the sum being taken over positive eigenvalues of A. The function ζ_A is well defined for s with a large enough real part (depending on the eigenvalues asymptotics), and has a meromorphic extension to the plane that is regular at $s = 0$. While (1.3) clearly equals the product of eigenvalues when there are only finitely many of them, the expression provides a regularisation of the otherwise divergent product. It is proven in [3] that

$$\det A = (-1)^N \det(\mathrm{Id}_{2N} - R(A)) \tag{1.4}$$

with $R(A)$ the monodromy operator. More precisely, $R(A)$ is equal to the fundamental matrix at time 1 of the linear time-varying system on $\mathrm{M}(2N, \mathbf{R})$

$$\begin{cases} \dot{R}(x) = \mathscr{A}_{V(x)} R(x), \\ R(0) = \mathrm{Id}_{2N}, \end{cases} \tag{1.5}$$

where one sets

$$\mathscr{A}_Q := \begin{bmatrix} 0 & \mathrm{Id}_N \\ Q & 0 \end{bmatrix}, \text{ for every } Q \in \mathrm{M}(N, \mathbf{R}).$$

Remark 1. In [3], the potential V appears as $-V$ in (1.5) and we have changed notations in order to stick with previous optimisation literature [1].

Since its trace is zero, the matrix $\mathscr{A}_V$ belongs to the lie algebra $\mathfrak{sl}(2N, \mathbf{R})$ and (1.5) defines a dynamics on the special linear group $\mathrm{SL}(2N, \mathbf{R})$, a Lie group of dimension $4N^2 - 1$. This dynamics is bilinear in R and V. We are now in

reference, however, provides much more elementary arguments enabling one to establish links with the discrete setting.

position to properly define the optimisation problems discussed in the present paper.

For every positive M, the set $\mathscr{V}_M$ of admissible Hill potentials is given by the measurable functions V so that

$$\mathscr{V}_M = \{V : [0,1] \to \mathrm{M}(N, \mathbf{R}) \mid \operatorname*{ess\,sup}_{x\in[0,1]} \|V(x)\| \leq M^2\}, \tag{1.6}$$

and we say that a potential V satisfies an L^∞-constraint if it belongs to some $\mathscr{V}_M$.

Remark 2. Note that $\mathscr{V}_M$ is convex and invariant by transposition and conjugation by orthogonal matrices, i.e. $V_{U(\cdot)} = U^T(\cdot)V(\cdot)U(\cdot)$ belongs to $\mathscr{V}_M$ if and only V does, for any measurable $\mathrm{SO}(N)$-valued $U(\cdot)$ defined on $[0,1]$. One could have defined equivalently $\mathscr{V}_M$ with potentials $V : \mathbf{R} \to \mathrm{M}(N, \mathbf{R})$ periodic of period 1 and satisfying the same L^∞ bound. In that case, $\mathscr{V}_M$ is clearly invariant by translation of $x_0 \in \mathbf{R}$, *i.e.* $V_{x_0}(\cdot) = V(\cdot + x_0)$ belongs to $\mathscr{V}_M$ if and only V does.

Remark 3. For $q \in [1, \infty)$, one could replace the L^∞ constraint by the integral condition

$$\int_0^1 \|V(x)\|^q \, \mathrm{d}x \leq M^{2q}$$

which is referred to as an L^q-constraint.

The cost function associated to a potential V is from now on denoted $\mathscr{C}(V)$ and is given by

$$\mathscr{C}(V) = (-1)^N \det\big(\mathrm{Id}_{2N} - R(1)\big), \tag{1.7}$$

where R is defined in (1.5). We will study the following optimisation questions: for every $M > 0$,

$$\mathbf{Max-Det(M)}: \quad \max_{V\in\mathscr{V}_M} \mathscr{C}(V) \text{ subject to (1.5)}, \tag{1.8}$$

$$\mathbf{Min-Det(M)}: \quad \min_{V\in\mathscr{V}_M} \mathscr{C}(V) \text{ subject to (1.5)}. \tag{1.9}$$

To derive common statements for both optimisation problems, we use $\mathscr{C}_\varepsilon$ to denote $\varepsilon\mathscr{C}$ where $\varepsilon = \pm 1$ and in that way **Max-Det** becomes the minimisation of $\mathscr{C}_-$ while **Min-Det** is simply the minimisation of $\mathscr{C}_+$. That is we study, for a given $M > 0$,

$$\mathbf{Ext-Det}_\varepsilon\mathbf{(M)}: \quad \min_{V\in\mathscr{V}_M} \mathscr{C}_\varepsilon(V) \text{ subject to (1.5)}. \tag{1.10}$$

This problem is a Mayer optimal control problem with state R in $\mathrm{SL}(2N, \mathbf{R})$, potential V (control) valued in a Euclidean ball of $\mathrm{M}(N, \mathbf{R})$, and bilinear dynamics. Control problems on Lie groups were intensively studied by Ivan Kupka and his collaborators [2], and were foundational for what has ever since emerged as *Geometric control theory.*

We begin our analysis in Section 1.2 by stating the necessary condition satisfied by optimisers (existence is clear). The problem can be formulated as an optimal control problem over the set of matrices with a matrix valued control, so the Pontryagin maximum principle provides the appropriate information. We also obtain some additional properties of optimisers Section 1.3. In Section 1.4 we focus on the one-dimensional case. We prove existence and uniqueness of maximisers and minimisers for the determinant over a bounded set in $L^\infty(\mathbf{S}^1)$.

1.2 Optimality conditions

In this section, we will derive the equations verified by the minimisers of $\mathbf{Ext}-\mathbf{Det}_\varepsilon$ as well as their first properties. From now on, M is an arbitrary positive number and $\varepsilon \in \{-1,1\}$. First of all, since $\mathscr{V}_M$ is non empty and, for any $R \in \mathrm{M}(2N,\mathbf{R})$, the set $\{\mathscr{A}_V \mid V \in \mathrm{M}(N,\mathbf{R}),\ \|V\| \leq M^2\}$ is compact and convex, then $\mathbf{Ext}-\mathbf{Det}_\varepsilon(\mathbf{M})$ admits minimisers according to Filippov theorem. According to the Pontryagin maximum principle (PMP), a solution R of $\mathbf{Ext}-\mathbf{Det}_\varepsilon(\mathbf{M})$ with minimising potential V is necessarily the projection of an *extremal*, *i.e.*, an integral curve $\lambda = (R,P) \in \mathrm{M}(2N,\mathbf{R})^2$ of a Hamiltonian vector field satisfying certain additional conditions. We hereby present a definition of extremal adapted to our setting. The fact that this is equivalent to the standard definition of normal extremal is the subject of Proposition 1 given below.

Definition 1. *A curve $\lambda : [0,T] \to \mathrm{M}(2N,\mathbf{R})^2$ is called* extremal *with respect to the control $V \in \mathscr{V}_M$ if:*

(i) Letting $\lambda = (R,P)$, it satisfies

$$\dot{R}(x) = \mathscr{A}_{V(x)} R(x), \tag{1.11}$$

$$\dot{P}(x) = -\mathscr{A}_{V(x)}^T P(x). \tag{1.12}$$

(ii) It holds that $R(0) = \mathrm{Id}_{2N}$ and the following transversality condition holds true[6]

$$P(1) = (-1)^N \varepsilon\ \mathrm{Com}\left(\mathrm{Id}_{2N} - R(1)\right). \tag{1.13}$$

(iii) Assume moreover that there exists $h \in \mathbf{R}$ such that a.e. on $[0,1]$

$$h = H\big(R(x), P(x), V(x)\big) = \max_{\|W\| \leq M^2} H\big(R(x), P(x), W\big), \tag{1.14}$$

where H is the Hamiltonian function defined on $\mathrm{M}(2N,\mathbf{R})^2 \times \mathrm{M}(N,\mathbf{R})$ by

$$H\big(R,P,W\big) = \langle P, \mathscr{A}_W R\rangle = \langle \mathscr{A}_W^T P, R\rangle. \tag{1.15}$$

such an extremal is called strong extremal.

[6] We denote $\mathrm{Com}(M)$ the comatrix of a square matrix M.

Remark 4. Note that every potential V admits a unique extremal (which is possibly strong).

We then get the following.

Proposition 1. *Let* $R : [0,T] \to \mathrm{M}(2N, \mathbf{R})$ *be an optimal trajectory of* $\mathbf{Ext} - \mathbf{Det}_\varepsilon(\mathbf{M})$ *with minimising potential* V*. Then* R *is the projection on* $\mathrm{M}(2N, \mathbf{R})$ *of a unique strong extremal* $\lambda = (R, P) : [0,T] \to \mathrm{M}(2N, \mathbf{R})^2$.

Proof. Let V be a minimising potential of $\mathbf{Ext} - \mathbf{Det}_\varepsilon(\mathbf{M})$ and R the associated trajectory by (1.5). Pontryagin maximum principle implies that there exists a nontrivial pair (p^0, P) where the cost multiplier p^0 is a nonpositive real number and the covector $P : [0,1] \to \mathrm{M}(2N, \mathbf{R})$ is a Lipschitz function so that

1. $(R(x), P(x)) \in \mathrm{M}(2N, \mathbf{R})^2$ satisfy on $[0,1]$ the adjoint equations:
$$\dot{R} = \nabla_P H, \tag{1.16}$$
$$\dot{P} = -\nabla_R H; \tag{1.17}$$
2. we have the maximality condition given by (1.14);
3. the following transversality condition holds true: $P(1) = p^0 \nabla \mathscr{C}_\varepsilon(V)$.

In addition, since H does not depend on time, its value in (1.14) does not depend on time and is denoted by the constant real number h. As

$$\nabla_P H = \mathscr{A}_W R, \quad \nabla_R H = \mathscr{A}_W^T P, \quad \nabla \det(\mathrm{Id}_{2N} - R) = -\operatorname{Com}(\mathrm{Id}_{2N} - R),$$

the items of Proposition 1 follow at once, except the facts that p^0 can be taken equal to -1 and λ is unique. To establish the first fact, it is enough to show that p^0 cannot be null. To show that, we argue by contradiction and, in that case, it follows that $P(1) = 0$. Since (1.19) is linear in P, one gets that P is identically equal to zero on $[0,1]$. This contradicts the non triviality of the pair (p^0, P) and hence $p^0 \neq 0$. Regarding the uniqueness of λ, note first that, given $M > 0$, trajectories of (1.5) are in one to one correspondence with potentials in $\mathscr{V}_M$, since to each such trajectory, there is a unique potential $V \in \mathscr{V}_M$ necessarily defined as the lower left $N \times N$ block of $\mathscr{A}_V = \dot{R}R^{-1}$ (recall that R is absolutely continuous). By Item 3., $P(1)$ is determined by $R(1)$ and hence P is computed from (1.17). □

To take advantage of the maximisation condition (1.14), after defining $q = PR^T$, we rewrite Proposition 1 only using q and we deduce at once that

Proposition 2. *Assume that a trajectory* R *of* $\mathbf{Ext} - \mathbf{Det}_\varepsilon(\mathbf{M})$ *with potential* V *is the projection of an extremal trajectory* $\lambda = (R, P)$*. Define*

$$q = PR^T = \begin{bmatrix} Z_1 & \psi \\ \varphi & Z_2 \end{bmatrix}, \tag{1.18}$$

where the various blocs are $N \times N$ *matrices. Then the dynamics of* q *is given, a.e. on* $[0,1]$*, by*[7]

[7] We denote $[Q_1, Q_2] = Q_1 Q_2 - Q_2 Q_1$ the commutator of matrices.

$$\dot q(x) = \left[q(x), \mathscr{A}^T_{V(x)}\right], \quad q(1) = (-1)^N \varepsilon \ \operatorname{Com}\big(\mathrm{Id}_{2N} - R(1)\big) R^T(1), \quad (1.19)$$

which yields, for a.e. $x \in [0,1]$,

$$\dot Z_1 = \psi - V^T\varphi, \quad (1.20)$$
$$\dot\varphi = Z_2 - Z_1, \quad (1.21)$$
$$\dot\psi = Z_1 V^T - V^T Z_2, \quad (1.22)$$
$$\dot Z_2 = \varphi V^T - \psi. \quad (1.23)$$

The Hamiltonian function H *defined in* (1.15) *is equal to*

$$H\big(R, P, W\big) = \langle q, \mathscr{A}^T_W\rangle = \operatorname{tr}(\psi) + \langle \varphi, W\rangle. \quad (1.24)$$

Moreover, it holds

$$q^T(x) = R(x) q^T(1) R^{-1}(x), \ \text{for every } x \in [0,1], \quad (1.25)$$
$$\ddot\varphi = -2\psi + V^T\varphi + \varphi V^T \ \text{for a.e. } x \in [0,1], \quad (1.26)$$

and in particular $q(\cdot)$ *is periodic of period one.*

Assume moreover $\lambda = (R,P)$ *is a strong extremal. If* $\varphi(x) \neq 0$, *then* $V(x) = M^2 \frac{\varphi(x)}{\|\varphi(x)\|}$ *and, for every* $x \in [0,1]$, *it holds*

$$h = \operatorname{tr}(\psi) + M^2\|\varphi(x)\|, \quad (1.27)$$
$$\operatorname{tr}(\ddot\varphi) = -2h + 4M^2\|\varphi(x)\|. \quad (1.28)$$

Proof. Most the above is immediate except (1.25). The latter follows from the fact that, for every $x \in [0,1]$,

$$q^T(x) = R(x)R^{-1}(1)q^T(1)R(1)R^{-1}(x).$$

The above equation then yields (1.25) after noticing that $R(1)$ and $q^T(1)$ commute. □

From now on, we indifferently call extremal either the pair (R,P) or the pair (R,q).

Remark 5. In the light of Item (*iii*) of the above proposition, one can see that the potential V is not (immediately) defined at a zero of φ. In the sequel, the latter function φ is refereed to as the *switching function* and we single out a particular instance of zero of φ, namely that of *switching time* defining such a point $x_* \in (0,1)$ for which $\varphi(x_*) = 0$ and there exist two sequences $(x_n)_{n\in\mathbf{N}}$ and $(y_n)_{n\in\mathbf{N}}$ of two by two distinct points, both converging to x_* such that $\langle\varphi(x_n), \varphi(y_n)\rangle < 0$ for $n \in \mathbf{N}$. Clearly, a zero of φ in $(0,1)$ which is not a zero of $\dot\varphi$ is a switching time.

Remark 6. At every $R \in \mathrm{SL}(2N, \mathbf{R})$, the tangent space is

$$T_R \,\mathrm{SL}(2N, \mathbf{R}) = \{rR \mid r \in \mathrm{M}(2N, \mathbf{R}) \text{ such that } \mathrm{tr}(r) = 0\}. \tag{1.29}$$

Using now the inner product introduced in (1.1), one can identify the cotangent space $T_R^* \,\mathrm{SL}(2N, \mathbf{R})$ as

$$T_R^* \,\mathrm{SL}(2N, \mathbf{R}) = \{q(R^{-1})^T \mid q \in \mathrm{M}(2N, \mathbf{R}) \text{ such that } \mathrm{tr}(q) = 0\}. \tag{1.30}$$

We next notice that the flow associated with (1.19) is isospectral (*cf.* for instance [5]), in particular the trace of q is constant on $[0, 1]$ equal to $\mathrm{tr}(q(1))$. Define indeed

$$\tilde{q}(x) = q(x) - \frac{\mathrm{tr}(q(1)}{2N} \,\mathrm{Id}_{2N}, \quad \tilde{P}(x) = \tilde{q}(x)\big(R^T(x)\big)^{-1}, \text{ for } x \in [0, 1].$$

Clearly the curve $\tilde{\lambda} = (R, \tilde{P})$ takes values in $T^* \,\mathrm{SL}(2N, \mathbf{R})$ and is an integral curve of the Hamiltonian vector field $\vec{H}$ associated with H. Finally, when applying the PMP to R, we claim that $\tilde{\lambda}$ turns out to be the required extremal with R as projection onto $\mathrm{SL}(2N, \mathbf{R})$: the dynamics of $\tilde{\lambda}$ has been described just previously, *i.e.*, $\dot{\tilde{\lambda}} = \vec{H}(\tilde{\lambda})$, the maximality condition is exactly (1.14) and the tranversality condition (1.13) now says that $P(1) - p^0 \nabla \mathscr{C}_\varepsilon(V)$ belongs to the normal cone at $T_{R(1)}^* \,\mathrm{SL}(2N, \mathbf{R})$, where the gradient is projected on $T_{R(1)}^* \,\mathrm{SL}(2N, \mathbf{R})$. Since that normal cone is equal to $\mathbf{R}(R^T(1))^{-1}$ and since one easily shows that $p^0 = -1$, one gets the claim regarding $\tilde{\lambda}$.

1.3 Invariance and symmetries

We begin by providing the following property regarding translated potentials ensuring that the problem is well posed for controls defined on $\mathbf{S}^1 \simeq \mathbf{R}/\mathbf{Z}$. In particular, the uniqueness results of Section 1.4 are stated for controls in $L^\infty(\mathbf{S}^1)$.

Lemma 1. *Let R be a trajectory of* $\mathbf{Ext} - \mathbf{Det}_\varepsilon(\mathbf{M})$ *associated with potential V and corresponding extremal (R, q). For $x_0 \in \mathbf{R}$, consider the potential V_{x_0} translated from V according to Remark 2. Then V_{x_0} has same cost as V with corresponding extremal (R_{x_0}, q_{x_0}) and one gets that*

$$q_{x_0}(x) = q(x + x_0), \quad \varphi_{x_0}(x) = \varphi(x + x_0), \quad \forall x \in \mathbf{R}. \tag{1.31}$$

where φ (φ_{x_0}, respectively) denotes the switching function associated with V (V_{x_0}, respectively).

Proof. It is immediate to derive that the trajectory R_{x_0} of (1.5) associated with V_{x_0} is given by

$$R_{x_0}(x) = R(x + x_0)R(x_0)^{-1}, \quad \forall x \in \mathbf{R}, \tag{1.32}$$

and, by periodicity of V, it follows that

$$R_{x_0}(1) = R(x_0)R(1)R(x_0)^{-1}. \tag{1.33}$$

Using the above equation, one gets that

$$\mathscr{C}_\varepsilon(V_{x_0}) = (-1)^N \varepsilon \det(\mathrm{Id}_{2N} - R_{x_0}(1)) = \mathscr{C}_\varepsilon(V),$$

and hence has same cost as V. Let $\lambda_{x_0} = (R_{x_0}, P_{x_0})$ be the unique extremal associated with R_{x_0}. Then, from (1.25), it holds

$$q_{x_0}^T(x) = R_{x_0}(x)\big(R_{x_0}(1)\big)^{-1} q_{x_0}^T(1) R_{x_0}(1)\big(R_{x_0}(x)\big)^{-1}, \quad \forall x \in [0,1],$$

and, from (1.19), one has

$$\begin{aligned}
q_{x_0}(1) &= (-1)^N \varepsilon \ \mathrm{Com}\left(\mathrm{Id}_{2N} - R_{x_0}(1)\right) R_{x_0}^T(1) \\
&= (-1)^N \varepsilon \ \mathrm{Com}\Big(\mathrm{Id}_{2N} - R(x_0)R(1)R(x_0)^{-1}\Big) R_{x_0}^T(1) \\
&= (-1)^N \varepsilon \ \big(R(x_0)^T\big)^{-1} \mathrm{Com}\left(\mathrm{Id}_{2N} - R(1)\right) R(x_0)^T R(x_0)^{-1} \\
&= \big(R(x_0)^T\big)^{-1} q(1) R(x_0)^T.
\end{aligned}$$

Using the above equation, (1.32) and (1.33), one gets (1.31). □

We then prove that there always exists potentials V with negative costs, implying that minimal values for $\mathbf{Ext} - \mathbf{Det}_\varepsilon(\mathbf{M})$ are always negative, which in particular, exclude the zero potential from optimality.

Lemma 2. *The cost $\mathscr{C}_\varepsilon(0)$ associated with the zero potential is equal to zero. For every $N \times N$ diagonal matrix $D = \mathrm{diag}(\varepsilon_1 d_1^2, \cdots, \varepsilon_N d_N^2)$, where $\varepsilon_i^2 = 1$ and $d_i > 0$ for $1 \le i \le N$, let $\mathscr{C}_\varepsilon(D)$ be the cost associated with the constant potential equal to D. Then*

$$\mathscr{C}_\varepsilon(D) = (-2)^N \varepsilon \Pi_{i=1}^N \big(1 - c_{\varepsilon_i}(d_i)\big). \tag{1.34}$$

Moreover, for every $M > 0$, $D \in \mathscr{V}_M$ if $\sum_{i=1}^N d_i^2 \le M^2$ and then $\mathscr{C}_\varepsilon(D) < 0$ if one chooses $\varepsilon_1 \varepsilon = -1$, $\varepsilon_i = 1$ for $2 \le i \le N$ and d_1 not a multiple of 2π if $\varepsilon_1 = -1$.

Proof. One clearly has that the trajectory R_0 of (1.5) associated with the zero potential is equal to

$$R_0(x) = \begin{bmatrix} \mathrm{Id}_N & x\,\mathrm{Id}_N \\ 0 & \mathrm{Id}_N \end{bmatrix} \text{ for } x \in [0,1].$$

The conclusion follows at once. Using (1.43), one easily deduces the value resolvent matrix R_D associated with D at $x = 1$,

$$R_D(1)=\begin{bmatrix}\operatorname{diag}(c_{\varepsilon_1}(d_1),\cdots,c_{\varepsilon_N}(d_N)) & \operatorname{diag}(\frac{s_{\varepsilon_1}(d_1)}{d_1},\cdots,\frac{s_{\varepsilon_N}(d_N)}{d_N})\\ \operatorname{diag}(\varepsilon_1 d_1 s_{\varepsilon_1}(d_1),\cdots,\varepsilon_N d_N s_{\varepsilon_N}(d_N)) & \operatorname{diag}(c_{\varepsilon_1}(d_1),\cdots,c_{\varepsilon_N}(d_N))\end{bmatrix}. \tag{1.35}$$

An elementary computation yields (1.34) and the lemma follows. □

We now derive basic facts on optimal trajectories.

Lemma 3. *Assume that R is an optimal trajectory associated with a minimising cost V and let h be the constant value of the Hamiltonian defined in* (1.14)*. Then the following facts hold true.*

(*a*) *The cost $\mathscr{C}_\varepsilon(V)$ is negative and hence $\operatorname{Id}_{2N}-R(1)$ is invertible. Moreover, the switching function φ is of class C^2, the matrix $q=RP^T$ defined in* (1.18) *is periodic of period one, it holds that*

$$q^T(1)=\mathscr{C}_\varepsilon(V)\big(\operatorname{Id}_{2N}-R(1)\big)^{-1}R(1) \text{ and } q^T(x)=R(x)q^T(1)R^{-1}(x) \tag{1.36}$$

for every $x\in[0,1]$, and the following relation holds true

$$h=2M^2\int_0^1\|\varphi(x)\|\,dx. \tag{1.37}$$

(*b*) *If $h=0$ then there exists an invertible $Z_*\in\mathrm{M}(N,\mathbf{R})$ such that*

$$q\equiv\begin{bmatrix}Z_* & 0\\ 0 & Z_*\end{bmatrix}, \tag{1.38}$$

and $(-1)^N\varepsilon$ is negative.

(*c*) *If $h>0$, then φ has a finite number of zeroes in $[0,1]$ at which either $\dot\varphi$ does not vanish or $\ddot\varphi$ is well defined and does not vanish.*

Proof. From (1.26) and the expression of V at points where φ does not vanish, one deduces that φ is of class C^2 on $[0,1]$. The one periodicity of q is an immediate consequence of Lemma 2. In that case, one can simplify (1.19) and (1.25) to get (1.36). The latter equation implies that $R(1)$ and $q^T(1)$ commute, which implies by using (1.36) that $q(0)=q(1)$. Since q is solution of a Cauchy problem (the ODE $\dot q=\left[q,\mathscr{A}_V^T\right]$ together with an initial condition), it follows that q is periodic of period one. Finally, integrating (1.28) between $x=0$ and $x=1$ and using the periodicity of $\operatorname{tr}(\dot\varphi)$, one gets (1.37). Assume $h=0$. From (1.37), it follows that $\varphi\equiv0$ and then (1.26) implies that $\psi\equiv0$ as well. The rest of the dynamics of q clearly yields that q is constant on $[0,1]$, verifying (1.38). By using the latter fact after taking the determinant in (1.36) it follows that

$$(\det Z_*)^2=(-1)^N\varepsilon\left[\det\big(\operatorname{Id}_{2N}-R(1)\big)\right]^{2N-1}=\mathscr{C}_\varepsilon(V)^{2N-1},$$

and the last part of Item (*b*) follows. We provide next an argument for Item (*c*). Arguing by contradiction, it would follow that there exists a sequence $(x_k)_{k\in\mathbf{N}}$

of two by two distinct times in $[0,1]$ so that $\lim_{k\to\infty} x_k = \bar{x}$ and $\varphi(x_k) = 0$ for $k \geq 0$. Since φ is of class C^1, one has that $\varphi(\bar{x}) = 0$ by continuity of φ and then

$$0 = \lim_{k\to\infty} \frac{\varphi(x_k) - \varphi(\bar{x})}{x_k - \bar{x}} = \dot{\varphi}(\bar{x}).$$

Since V is bounded, one deduces from (1.26) that $\ddot{\varphi}$ is twice differentiable at $x = \bar{x}$. Moreover, $\ddot{\varphi}(\bar{x})$ is not zero since, from (1.26), it holds

$$\operatorname{tr}(\ddot{\varphi}(\bar{x})) = -2h < 0.$$

By a Taylor expansion at order two, one obtains that there exists an open interval I centered at $\bar{x}$ so that $\varphi(x) = 0$ for $x \in I$ only if $x = \bar{x}$. That contradicts the existence of the sequence $(x_k)_{k\in\mathbf{N}}$. □

We end the section by providing preliminary symmetry properties for a minimising potential. For that purpose we define the following matrices of $\mathrm{M}(2N,\mathbf{R})$:

$$J = \mathscr{A}_{\mathrm{Id}_{2N}} \text{ i.e. } J = \begin{bmatrix} 0 & \mathrm{Id}_N \\ \mathrm{Id}_N & 0 \end{bmatrix}, \quad A = \mathscr{A}_{-\mathrm{Id}_{2N}} \text{ i.e. } A = \begin{bmatrix} 0 & \mathrm{Id}_N \\ -\mathrm{Id}_N & 0 \end{bmatrix},$$

$$\mathscr{U} = \begin{bmatrix} U & 0 \\ 0 & U \end{bmatrix}, \text{ for every } U \in \mathrm{SO}(N), \quad \mathscr{B}_Q = \mathscr{A}^T_{Q^T}, \text{ for every } Q \in \mathrm{M}(N,\mathbf{R}).$$

Note that $J^2 = A^T A = \mathrm{Id}_{2N}$.

Proposition 3. *Let $M > 0$, $V \in \mathscr{V}_M$ and R the trajectory of* (1.5) *associated with V. The following items are equivalent:*

(1.) *V is a minimising potential for $\mathbf{Ext} - \mathbf{Det}_\varepsilon(M)$ along* (1.5)*;*
(2.) *for every $U \in \mathrm{SO}(N)$, $V_{\mathscr{U}} = \mathscr{U}^T V \mathscr{U}$ is a minimising potential for $\mathbf{Ext} - \mathbf{Det}_\varepsilon(M)$ along* (1.5) *with $\mathscr{U}^T R \mathscr{U}$ as associated trajectory;*
(3.) *V is a minimising potential for $\mathbf{Ext} - \mathbf{Det}_\varepsilon(M)$ along trajectories of each of the following four dynamical systems*

$$\begin{cases} \dot{S}(x) = \mathscr{B}_{V(x)} S(x), \\ S(0) = \mathrm{Id}_N, \end{cases} \quad \begin{cases} \dot{S}(x) = -\mathscr{B}_{V(x)} S(x), \\ S(0) = \mathrm{Id}_N, \end{cases}$$

$$\begin{cases} \dot{S}(x) = S(x)\mathscr{B}_{V^T(x)}, \\ S(0) = \mathrm{Id}_N, \end{cases} \quad \begin{cases} \dot{S}(x) = -S(x)\mathscr{A}_{V(x)}, \\ S(0) = \mathrm{Id}_N, \end{cases}$$

with JRJ, $A^T RA$, R^T and R^{-1} as associated optimal trajectories respectively and same value of the cost;
(4.) *V^T is a minimising potential for $\mathbf{Ext} - \mathbf{Det}_\varepsilon(M)$ along* (1.5) *with associated trajectory $A^T(R^T)^{-1}A$.*

Proof. Showing the several items is immediate once one notices that

$$J\mathscr{A}_Q J = \mathscr{B}_Q, \quad A^T \mathscr{A}_Q A = -\mathscr{B}_Q, \text{ for every } Q \in M_N(\mathbf{R}).$$

As for the equality of the costs, we just check the following

$$\det(\mathrm{Id}_{2N} - R^{-1}(1)) = \det\Big((R(1) - \mathrm{Id}_{2N})R^{-1}(1)\Big) = \det(\mathrm{Id}_{2N} - R(1)).$$

□

1.4 One-dimensional case

From now on $N = 1$, M is still a positive number and

$$\mathscr{V}_M = \{V : [0,1] \to \mathbf{R} \mid V \text{ measurable and } \operatorname*{ess\,sup}_{x\in[0,1]} |V(x)| \le M^2\}. \quad (1.39)$$

From Item *(iii)* of Proposition 2, it holds that $V(x) \in \{-M^2, M^2\}$ as soon as $\varphi(x) \neq 0$ and this motivates the following definition.

Definition 2. *Let R be a trajectory of* (1.5) *associated to some $V \in \mathscr{V}_M$. A* bang arc *$\gamma : I \to \mathrm{M}(2,\mathbf{R})$ is a piece of R defined on some non empty subinterval $I \subset [0,1]$ such that $V = \nu M^2$ is constant on I, with $\nu \in \{-1, 1\}$. A trajectory R of* (1.5) *is said to be* bang *if it is made of a unique bang arc and* bang-bang *if it is the concatenation of bang arcs.*

We first examine the **Max-Det** problem. In dimension $N = 1$, the cost to maximise is

$$\begin{aligned}\mathscr{C}_V &= -\det(I_2 - R(1)),\\ &= -(1 - \operatorname{tr} R(1) + \det R(1)),\\ &= \operatorname{tr} R(1) - 2\end{aligned}$$

since the monodromy $R(1)$ belongs to $\mathrm{SL}(2,\mathbf{R})$. Maximising $\mathscr{C}_V$ is so equivalent to maximising the trace of the monodromy

$$\operatorname{tr} R(1) = z(1) + y'(1),$$

where z and y satisfy $-w'' + V(x)w = 0$ with respective initial conditions $(z(0), z'(0)) = (1, 0)$ and $(y(0), y'(0)) = (0, 1)$.

Proposition 4. *Let V_1 and V_2 be two potentials in $L^1_{\mathrm{loc}}(\mathbf{R}_+)$, $V_1 \ge |V_2|$ a.e., and let y_1 and y_2 satisfy $-y_i'' + V_i(x)y_i = 0$, $i = 1, 2$. If $y_1(0) \ge |y_2(0)|$ and $y_1'(0) \ge |y_2'(0)|$, then $y_1(x) \ge |y_2(x)|$ and $y_1'(x) \ge |y_2'(x)|$ for all $x \ge 0$.*

Proof. (i) First assume V_1 and V_2 constant, $V_1 \equiv A$ and $V_2 \equiv B$ with A and B two reals such that $A \ge |B|$. One has

$$y_1(x) = y_1(0)\cosh(\alpha x) + x y_1'(0)\,\mathrm{sinhc}(\alpha x)$$

where $\alpha = \sqrt{A}$, and where we denote

$$\operatorname{sinhc}(x) = \begin{cases} \sinh(x)/x & \text{if } x \neq 0, \\ 1 & \text{if } x = 0. \end{cases}$$

If B is nonnegative, let $\beta := \sqrt{B} \leq \alpha$; one has

$$\begin{aligned} |y_2(x)| &= |y_2(0)\cosh(\beta x) + x y_2'(0)\operatorname{sinhc}(\beta x)| \\ &\leq |y_2(0)|\cosh(\beta x) + x|y_2'(0)|\operatorname{sinhc}(\beta x) \\ &\leq y_1(0)\cosh(\alpha x) + x y_1'(0)\operatorname{sinhc}(\alpha x) = y_1(x) \end{aligned}$$

for $x \geq 0$ since both cosh and sinhc are nondecreasing functions on $\mathbf{R}_+$ (and $\beta \leq \alpha$). Similarly, for $x \geq 0$,

$$\begin{aligned} |y_2'(x)| &= |\beta y_2(0)\sinh(\beta x) + y_2'(0)\cosh(\beta x)| \\ &\leq \alpha y_1(0)\sinh(\alpha x) + y_1'(0)\cosh(\alpha x) = y_1'(x). \end{aligned}$$

If B is negative, let $\beta := \sqrt{-B} \leq \alpha$; one has (denoting $\operatorname{sinc}(x) = \sin(x)/x$ if $x \neq 0$, $\operatorname{sinc}(0) = 1$)

$$\begin{aligned} |y_2(x)| &= |y_2(0)\cos(\beta x) + x y_2'(0)\operatorname{sinc}(\beta x)| \\ &\leq |y_2(0)|\cosh(\beta x) + x|y_2'(0)|\operatorname{sinhc}(\beta x) \\ &\leq y_1(x) \end{aligned}$$

for $x \geq 0$ since $|\cos| \leq \cosh$ and $|\operatorname{sinc}| \leq \operatorname{sinhc}$ on $\mathbf{R}_+$. Similarly, for $x \geq 0$,

$$\begin{aligned} |y_2'(x)| &= |-\beta y_2(0)\sin(\beta x) + y_2'(0)\cos(\beta x)| \\ &\leq \alpha y_1(0)\sinh(\alpha x) + y_1'(0)\cosh(\alpha x). \end{aligned}$$

(ii) Take now some positive x, and assume V_1 and V_2 are piecewise constant on $[0, x]$; there exists a common subdivision $0 = x_0 < x_1 < ... < x_N = x$, $N \geq 1$, such that on every $[x_i, x_{i+1}[$ both V_1 and V_2 are constant, with $V_1 \geq |V_2|$. A simple recurrence using step (i) allows to conclude that $y_1(x) \geq |y_2(x)|$ and $y_1'(x) \geq |y_2'(x)|$.

(iii) Consider eventually V_1 and V_2 locally integrable on $\mathbf{R}_+$, and fix $x > 0$. There exist two sequences $(V_{1,n})_n$, $(V_{2,n})_n$ of piecewise constant functions converging in $L^1(0, x)$ to V_1 and V_2, respectively. These sequences can be chosen such that $V_{1,n} \geq |V_{2,n}|$, $n \in \mathbf{N}$. Then according to point (ii), for all $n \in \mathbf{N}$, $y_{1,n}(x) \geq |y_{2,n}(x)|$ and $y_{1,n}'(x) \geq |y_{2,n}'(x)|$, where $y_{i,n}$ denotes the solution associated with $V_{i,n}$ and fixed initial conditions $(y_i(0), y_i'(0))$, $i = 1, 2$. Since, for any given initial condition (y_0, y_0'), the mapping $V \mapsto (y(x), y'(x))$ (where y is the solution of $-y'' + Vy = 0$, $y(0) = y_0$, $y'(0) = y_0'$) is continuous from $L^1(0, x)$ to $\mathbf{R}^2$ (see, *e.g.*, Proposition 7 in [1]), passing to the limit one obtains that $y_1(x) \geq |y_2(x)|$ and $y_1'(x) \geq |y_2'(x)|$. As x is arbitrary, the desired conclusion holds. □

Corollary 1. *For V in $L^\infty(0,1)$, let y and z denote the solutions of*

$$-y'' + V(x)y = 0, \quad y(0) = 0, \quad y'(0) = 1,$$

$$-z'' + V(x)z = 0, \quad z(0) = 1, \quad z'(0) = 0.$$

Then, for any positive bound M, the constant potential $V \equiv M^2$ is the unique function maximising both $y(1)$, $y'(1)$, $z(1)$ and $z'(1)$ over essentially bounded potentials such that $\|V\|_\infty \leq M^2$.

Theorem 1. *The unique solution of the* **Max-Det(M)** *problem in the periodic case is the constant potential equal to M^2.*

Proof. It is clear from the previous corollary that the constant potential $V \equiv M^2$ maximises $z(1) + y'(1)$ among essentially bounded potentials such that $\|V\|_\infty \leq M^2$. Let V be a measurable function satisfying the same bound and such that V is strictly inferior to M^2 on a positive measure subset of $[0,1]$; a direct estimation allows to prove that the associated values of both $z(1)$ and $y'(1)$ (hence of their sum) are strictly smaller than the values obtained for the constant potential $V \equiv M^2$. □

We eventually handle **Min-Det**. In particular, we immediately derive the following result after Lemmas 3 and 2.

Lemma 4. *Assume that R is an optimal trajectory associated with a potential V minimising $\mathscr{C}_1$. Then the following cases may occur.*

(i) If $h = 0$, then V is equal to the constant potential $V_0 \equiv -M^2$ and φ never vanishes on on $[0,1]$. In that case, the minimal cost is equal to $\mathscr{C}_1(V_0) = -2\big(1 - c_-(M)\big)$;
(ii) if $h \neq 0$, then φ has a finite number of zeroes in $[0,1]$ and $V(x) = M^2\mathrm{sgn}(\varphi(x))$ outside a finite set made of the zeroes of φ.

Hence, either R is the bang trajectory R_0 associated with V_0 or it is a bang-bang trajectory with a finite number of bang arcs.

Proof. From Lemma 2, we know that the minimal value of $\mathscr{C}_1$ is negative, and then, Item (a) of Lemma 3 only leaves the possibility of φ never vanishing on $[0,1]$. Hence V is constant equal M or $-M$. Since $C_1(M) > 0$, Item (i) follows at once. Item (ii) is essentially a rewriting of Item (b) of Lemma 3 together with Item (iii) of Proposition 2. □

In the one-dimensional case, we can actually give a more elementary proof that minimising potentials are bang-bang with finitely many switchings using the structure of $\mathfrak{sl}(2,\mathbf{R})$. Our minimisation problem is a Mayer problem with linear cost, $\mathrm{tr}\, R(1) \to \min$, and bilinear dynamics

$$\dot{R}(x) = F_0 R(x) + V(x) F_1 R(x)$$

with a single input control such that, a.e., $|V(x)| \leq M^2$, and matrices (linear vector fields)

$$F_0 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad F_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Together with their commutator[8]

$$F_{01} := [F_0, F_1] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

these matrices form an $\mathfrak{sl}_2$-*triple* of the dimension three Lie algebra. In particular, one has

$$F_{001} = [F_0, F_{01}] = -2F_0, \quad F_{101} = [F_1, F_{01}] = 2F_1. \tag{1.40}$$

Denoting $H_i := \langle P, F_i R\rangle$, for $i = 0, 1$, the Hamiltonian lifts of F_0 and F_1, the Hamiltonian is $H = H_0 + VH_1$. To analyse the structure of the set of zeroes of H_1 along an extremal, one can compute (with the same notation as before)

$$\dot{H}_1 = H_{01}, \quad \dot{H}_{01} = H_{001} + VH_{101}.$$

Because of (1.40), $\ddot{H}_1 = 2(VH_1 - H_0)$ so H_1 is $\mathscr{C}^2$ (since V is bounded, VH_1 vanishes whenever H_1 does) and there are two cases at a switching time: either H_{01} is not zero, or H_{01} is zero and H_{001} is not (P would otherwise vanish, which is forbidden, since F_0, F_1 and F_{01} form a basis of the Lie algebra). In both cases, the switching time must be isolated.

We focus now on strong extremals associated with $h \neq 0$, and introduce the following notations: if $\nu^2 = 1$, we use $c_\nu(t)$ (respectively $s_\nu(t)$) to denote $\cosh(t)$ if $\nu = 1$ and $\cos(t)$ if $\nu = -1$ (respectively $\sinh(t)$ if $\nu = 1$ and $\sin(t)$ if $\nu = -1$). With these conventions, one also has for every $x \in \mathbf{R}$ that

$$c_\nu^2(x) - \nu s_\nu^2(x) = 1,\ \dot{c}_\nu(x) = \nu s_\nu(x),\ \dot{s}_\nu(x) = c_\nu(x), \tag{1.41}$$

$$c_\nu(2x) = 1 + 2\nu s_\nu^2(x),\ s_\nu(2x) = 2\nu s_\nu(x) c_\nu(x). \tag{1.42}$$

As a consequence, if d is a positive real number, the solution of the linear second order equation $\ddot{y} = \nu d y$ is given by

$$y(t) = c_\nu(dt) y(0) + \frac{1}{d} s_\nu(dt) \dot{y}(0), \quad t \in \mathbf{R}. \tag{1.43}$$

We have the following two intermediate results.

Lemma 5. *Let (R, q) be a strong extremal projecting on an optimal trajectory R which is associated with a potential V minimising $\mathscr{C}_1$ with corresponding $h \neq 0$. Assume furthermore that*

[8] Note that we use the matrix commutator whose sign is opposite to the Lie bracket of the associated linear vector fields.

1. *V is not identically equal to $-M^2$;*
2. *$x_0 < x_1$ are two consecutive zeroes of φ in $[0,1]$,* i.e., *$|\varphi| > 0$ on (x_0, x_1).*

Set $T := x_1 - x_0 > 0$ and $\nu = \operatorname{sgn}(\varphi)$ on (x_0, x_1). Then both $c_\nu(MT)$ and $s_\nu(MT)$ are non zero and the following holds:

$$\varphi(x) = \frac{h}{M^2 c_\nu(MT)} s_\nu(M(x - x_0)) s_\nu(M(x_1 - x)), \text{ for } x \in [x_0, x_1]. \quad (1.44)$$

In particular,

$$\dot{\varphi}(x_0) = -\dot{\varphi}(x_1) = h \frac{s_\nu(MT)}{c_\nu(MT)} \neq 0. \quad (1.45)$$

Proof. In the case $N = 1$ and using the notations of the lemma, one can rewrite (1.26) as

$$\ddot{\varphi} = 4\nu M^2 (\varphi - \frac{\nu h}{2M^2}) \text{ for } x \in [x_0, x_1]. \quad (1.46)$$

Integrating (1.46) yields that

$$\varphi(x) = \frac{\nu h}{2M^2}\big(1 - c_\nu(2M(x - x_0))\big) + B s_\nu(2M(x - x_0)), \quad (1.47)$$

$$\dot{\varphi}(x) = 2M^2 B c_\nu(2M(x - x_0)) - h s_\nu(2M(x - x_0)), \quad (1.48)$$

where B is a constant satisfying

$$-\frac{\nu h}{2M^2}(1 - c_\nu(2MT)) = B s_\nu(2MT). \quad (1.49)$$

From (1.48), one deduces that

$$\dot{\varphi}(x_0) = 2M^2 B, \quad \dot{\varphi}(x_1) = 2M^2 B c_\nu(2MT) - h s_\nu(2MT). \quad (1.50)$$

We prove next that $s_\nu(MT) \neq 0$. Arguing by contradiction, it would first imply that $\nu = -1$ and then $V = -M^2$, $c_\nu(2MT) = 1$, $s_\nu(2MT) = 0$ and, from (1.50), that $\dot{\varphi}(x_0) = \dot{\varphi}(x_1) = 2M^2 B$. If $B \neq 0$, then $\operatorname{sgn}(B)\dot{\varphi}$ is positive in a right neighborhood of x_0 while it is negative in a left neighborhood of x_1, implying that φ must vanish inside (x_0, x_1). This contradicts Item 2., and therefore one deduces that $B = 0$ and then $\ddot{\varphi}(x_0) = \ddot{\varphi}(x_1) = -2h$, yielding that $h > 0$ and x_0 and x_1 are not switching times. We claim that every zero of φ is not a switching time and that $V \equiv -M^2$. Indeed, recall that a zero of φ is isolated and there are a finite number of them. Consider then x_2 distinct from x_0 and x_1. Assume that it is consecutive to x_1, *i.e.* $|\varphi| > 0$ on (x_1, x_2). Reproducing the reasoning done on $[x_0, x_1]$ with x_1 (respectively x_2) replacing x_0 (respectively x_1), we conclude from (1.50) that the corresponding B is equal to zero and from (1.47) that $c_{\nu'}(2M(x_2 - x_1)) = 1$, *i.e.*, $\nu' = -1$ and $s_{\nu'}(M(x_2 - x_1)) = 0$. Being back to the previous situation, one deduces that $\dot{\varphi}(x_2) = 0$. Proceeding in that way step by step, one gets the claim. This contradicts Item 1. and finally one has proved that $s_\nu(MT) \neq 0$. From (1.49) and (1.42), one gets that

$$B = \frac{h}{2M^2} \frac{c_\nu(MT)}{s_\nu(MT)},$$

and direct computations finally yield (1.44) and (1.46). □

To state our subsequent results, one needs to define, for every positive real number M the function $F_M : [0,1] \to \mathbf{R}_+$ by

$$F_M(x) = x + \frac{\pi - \arctan\big(\tanh(Mx)\big)}{M}. \tag{1.51}$$

The basic facts on this function are the following:

$$F_M(0) = \frac{\pi}{M},\ F_M(1) = 1 + \frac{\pi - \arctan\big(\tanh(M)\big)}{M}, F_M'(x) = \frac{2\tanh^2(Mx)}{1+\tanh^2(Mx)}, \tag{1.52}$$

for all $x \in [0,1]$. Hence F_M is a C^1, strictly increasing bijection from $[0,1]$ to $[\frac{\pi}{M}, F_M(1)]$ and $F_M(1) > 1$. Our second intermediate result goes as follows.

Lemma 6. *Let (R,q) be a strong extremal projecting on an optimal trajectory R which is associated with a potential V minimising $\mathscr{C}_1$ with corresponding $h \neq 0$. Assume furthermore that R is not a bang trajectory. Then, up to a translation, V is periodic of period $T_1 + T_2$ so that $V = M^2$ on $[0,T_1]$ and $V = -M^2$ on $[T_1, T_1+T_2]$ where $T_1, T_2 \in (0,1)$ so that they satisfy*

$$T_2 = \frac{\pi - \arctan\big(\tanh(MT_1)\big)}{M}, \tag{1.53}$$

and there exists a positive integer l such that

$$F_M(T_1) = 1/l. \tag{1.54}$$

Proof. Notice that R must have at least two distinct bang arcs and then at least two switching points. Moreover, all the zeroes of φ must be switching times according to (1.45). Thanks to Lemma 1, we can assume, up to translating the potential V, that 0 is a switching time and $\varphi > 0$ in a right neighborhood of zero (since both signs are taken on $[0,1]$). Since $\dot\varphi(0) \neq 0$, it must be positive and (1.45) yields that both h and ν are positive. We first claim that $x = 1$ must be a switching time. For otherwise, $\varphi(1) \neq 0$ and hence V has a constant sign in a left neighborhood of 1. If $V = M^2$ there, then for $a > 0$ small enough one has that $\varphi_{-a}(a) = \varphi(0) = 0$ and $\dot\varphi_{-a}(a) = \dot\varphi(0) \neq 0$, *i.e.*, a is a switching time for V_{-a}. This is in contradiction with the fact that $V_{-a} = M$ in an open neighborhood of a. If now $V = -M^2$ in a left neighborhood of 1, let $x_r < 1$ be the largest zero of φ in $[0,1]$. It turns out that V_{x_r} changes sign at $x = 1 - x_r$

but this is in contradiction with the fact that $\varphi_{x_r}(1-x_r)=\varphi(1)\neq 0$. We have proved the claim. Now we show that the last bang must correspond to $V=-M^2$. Indeed if it were not the case, then $V_a=M^2$ in an open neighborhood of some $a>0$ small enough with $\varphi_a(a)=0$, which is not possible. It means that R is the concatenation of an even number of bang arcs, γ_i, $1\leq i\leq 2l$, where on the γ_{2j-1}'s, $1\leq j\leq l$, one has $V=M^2$ and on the γ_{2j}'s, $1\leq j\leq l$, one has $V=-M^2$. Let $T_i>0$ be the duration of each bang arc γ_i, for $1\leq i\leq 2l$, and clearly

$$\sum_{i=1}^{2l} T_i = 1. \tag{1.55}$$

We next prove that $T_2=F(T_1)$. Indeed, consider (1.45) written for $(x_0,x_1)=(0,T_1)$ and then $(x_0,x_1)=(T_1,T_1+T_2)$. One deduces that

$$h\tanh(MT_1)=\dot\varphi(0)=-\dot\varphi(T_1),\quad h\tan(MT_2)=\dot\varphi(T_1)=-\dot\varphi(T_1+T_2). \tag{1.56}$$

It follows at once that

$$\tanh(MT_1)=-\tan(MT_2)\in(0,1).$$

It follows that $MT_2-k\pi\in(\frac{3\pi}{4},\pi)$ for some non negative integer k. Then $k=0$ otherwise, using (1.44), φ would have another zero in (T_1,T_1+T_2), which is not possible. One deduces (1.53). We finally prove that

$$T_{2j-1}=T_1,\quad T_{2j}=T_2,\ \text{for } 1\leq j\leq l. \tag{1.57}$$

We only provide an argument for $T_3=T_1$ since the other equalities are deduced in an identical manner. For that purpose, consider (1.45) written for $(x_0,x_1)=(T_1+T_2,T_1+T_2+T_3)$. One deduces that

$$h\tanh(MT_3)=\dot\varphi(T_1+T_2)=-\dot\varphi(T_1+T_2+T_3).$$

Using (1.56), one gets that

$$\tanh(MT_3)=\frac{\dot\varphi(T_1+T_2)}{h}=-\tan(MT_2)=\tanh(MT_1),$$

yielding that $T_1=T_3$ and V is (T_1+T_2)-periodic. One deduces (1.54) from (1.55), which concludes the proof of Lemma 6. □

We are able to state the proposition providing a complete solution to **Min-Det** in the case $N=1$.

Theorem 2. *For every positive M, the optimal control problem* **Min-Det(M)** *admits a unique minimising potential V_{min} in $L^\infty(\mathbf{S}^1)$ defined as follows.*

(*a*) *If $M\in(0,\pi]$, $V_{min}=V_0\equiv -M^2$ and the minimal value for* **Min-Det(M)** *is equal to $\mathscr{C}_1(V_0)=-2\big(1-c_-(M)\big)$;*

(*b*) *If* $M > \pi$*,* V_{min} *is the potential* V_1 *equal to* M^2 *on* $[0, t_1]$ *and* $-M^2$ *on* $[t_1, 1]$*, with* $F_M(t_1) = 1$ *and the minimal value for* **Min-Det(M)** *is equal to* $\mathscr{C}_1(V_1) = -2\big(1 - c_-(M(1-t_1))c_+(t_1)\big)$*.*

Proof. If $M \leq \pi$, then $F_M(x) > 1$ for every $x \in (0,1]$ and one deduces from (1.54) that there is no $T_1 \in (0,1)$ satisfying the properties required for the existence of a an optimal trajectory R which is not a bang trajectory. Therefore, the only candidate left as minimising potential by Lemma 4 is $V = V_0$, *i.e.* Item (a) holds true. Assume now that $M > \pi$. Define the positive integer $L := E(\frac{M}{\pi})$ (where $E(x)$ stands for the integer part of the real x), and the $2L$ times

$$t_l = F_M^{-1}(1/l),\ s_l = 1/l - t_l, \quad 1 \leq l \leq L. \tag{1.58}$$

According to Lemma 6, there exists a bang-bang trajectory R_l with $2l$ bang arcs and associated with the periodic potential V_l of period $1/l$ so that $V_l = M^2$ on $[0, t_l]$ and $V_l = -M^2$ on $[t_l, t_l + s_l]$. Recall that R_0 is the trajectory of (1.5) associated with V_0. Then, one gets from Lemmas 4 and 6 that a minimising potential V_{min} must be equal to V_l for some integer $0 \leq l \leq L$. In order to conclude, one is left with the computation of the costs $\mathscr{C}_1(V_l)$, for positive integers $1 \leq l \leq L$. A lengthy but straightforward computation yields that

$$\begin{aligned} R_l(1/l) &= \begin{bmatrix} c_-(Ms_l) & \frac{s_-(Ms_l)}{M} \\ -Ms_-(Ms_l) & c_-(Ms_l) \end{bmatrix} \begin{bmatrix} c_+(Mt_l) & \frac{s_+(Mt_l)}{M} \\ Ms_+(Mt_l) & c_+(Mt_l) \end{bmatrix} \\ &= \begin{bmatrix} c_-(Ms_l)c_+(Mt_l) + s_-(Ms_l)s_+(Mt_l) & \frac{c_-(Ms_l)s_+(Mt_l) + s_-(Ms_l)c_+(Mt_l)}{M} \\ M(s_-(Ms_l)c_+(Mt_l) + c_-(Ms_l)s_+(Mt_l)) & c_-(Ms_l)c_+(Mt_l) - s_-(Ms_l)s_+(Mt_l) \end{bmatrix}, \end{aligned} \tag{1.59}$$

and one has that $\det(R_l(1/l)) = 1$ and

$$\alpha_l = -\frac{\operatorname{tr}(R_l(1/l))}{2} = -c_-(Ms_l)c_+(Mt_l),\ 1 \leq l \leq L. \tag{1.60}$$

We use $r_l, \frac{1}{r_l}$ in $\mathbf{C}$ to denote the eigenvalues of $R_l(1/l)$. Since V_l is $1/l$-periodic, one gets that $R_l(1) = R_l^l(1/l)$ and hence

$$\mathscr{C}_1(V_l) = -\det\big(\mathrm{Id}_2 - R_l^l(1/l)\big) = (-2)(1 - \frac{r_l^l + r_l^{-l}}{2}),\ 1 \leq l \leq L. \tag{1.61}$$

Recall that $Ms_l \in (\frac{3\pi}{4}, \pi)$ and hence, it holds, for $1 \leq l \leq L$ that

$$\begin{aligned} -c_-(Ms_l) &= -c_-\Big(\pi - \arctan\big(\tanh(Mt_l)\big)\Big) = c_-\Big(\arctan\big(\tanh(Mt_l)\big)\Big) \\ &= \frac{1}{\sqrt{1 + \tanh^2(Mt_l)}} = \frac{c_+(Mt_l)}{\sqrt{c_+^2(Mt_l) + s_+^2(MT_1)}}, \end{aligned}$$

and then

$$\alpha_l = \frac{c_+^2(Mt_l)}{\sqrt{2c_+^2(Mt_l) - 1}} > 1. \tag{1.62}$$

Let $\xi_l > 0$ such that $\alpha_l = c_+(\xi_l)$. Since r_l and $\frac{1}{r_l}$ are the roots of the degree two polynomial $X^2 + 2c_+(\xi_l)X + 1$, one gets that $r_l = -e^{\xi_l}$ and finally it holds

$$\mathscr{C}_1(V_l) = (-2)\Big(1 - (-1)^l c_+(l\xi_l)\Big).$$

For even l's, the cost is non negative, implying that V_l cannot be minimising. For odd l's, the cost is smaller than -4 and then smaller than $\mathscr{C}_1(V_0)$. It remains to show that $\mathscr{C}_1(V_l)$ reaches its minimal value for $l = 1$. For that, it is enough to prove that the mapping $G : l \mapsto l\xi_l$ is strictly decreasing for $l \in [1, L]$. Computing, one gets

$$G'(l) = Mt_l\Big(\frac{\xi_l}{Mt_l} - \frac{c_+(Mt_l)}{s_+(Mt_l)}\frac{F_M(Mt_l)}{t_l}\Big), \quad l \in [1, L].$$

Since $F_M(Mt_l) > t_l$, one would have that $G'(l) < 0$ if one shows that $\xi_l < Mt_l$. In turn, that last inequality is itself equivalent $\alpha_l < c_+(Mt_l)$, inequality which does hold true by (1.62). This concludes the proof of Theorem 2. □

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