Numerical optimal control and orbital transfers

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Abstract

The transfer of a satellite around the Earth is modelled by the controlled Kepler equation. The existence and the structure of solutions is studied geometrically while several performance indexes are considered, namely minimization of the transfer time and maximization of the final mass. For each of these, homotopic techniques are used: discrete continuation, predictor-corrector or piecewise linear methods. Results are given for very low thrusts (*e.g.* 0.1 Newton).

Introduction

We address the problem of transfering a satellite with a low-thrust engine from an eccentric initial orbit to a higher geostationnary final one. The minimum time transfer has been quite extensively studied, we refer to [8, 11, 2, 7].

Hence, after a brief recall of the mathematical model in the first section, of its main geometric properties and of the main results in the min time case, respectively in the second and third section, the last section will be devoted to the newest material, namely the maximization of the mass.

In what follows, we will try to illustrate to what extent a geometric point of view in the spirit of [10] is relevant in this context, and also to show how homotopic techniques in the broad sense (discrete, continuous, simplicial...) are ubiquitous in this work.

Transfer problem

The mathematical model is the controlled Kepler equation. Indeed, we neglect second order terms of the Earth potential, so that the equations write:

$$\ddot{r} = -kr/|r|^3 + u/m \qquad (1)$$
$$\dot{m} = -\beta|u|. \qquad (2)$$

where the position vector r and the control u (the thrust of the engine) are valued in $\mathbf{R}^3$, whereas m, the mass, is in $\mathbf{R}$. As in (1-2), $|.|$ will denote Euclidean norms throughout the text: $|u| = (|u_1|^2 + |u_2|^2 + |u_3|^2)^{1/2}$. The dynamics in r is easily restated as an ordinary differential equation on the open submanifold X of $\mathbf{R}^6$ defined by

$$r \neq 0,\ 1/2|\dot{r}|^2 - k/|r| < 0$$

while the mass is of course prescribed to remain positive, $m > 0$. As before, the positive constant k stands for the attraction of the Earth. Adding the maximum thrust constraint

$$|u_1|^2 + |u_2|^2 + |u_3|^2 \leq T_{max}^2$$

together with boundary conditions

$$r(0),\ \dot r(0),\ m(0) \text{ fixed}$$
$$h(r(t_f), \dot r(t_f)) = 0$$

we consider two criterions: the minimization of the (free) transfer time, t_f, and the maximization of the final mass, $m(t_f)$ (in the last case, the transfer time will be assumed to be fixed and strictly greater than the minimal one). An outline of the main properties of the control system is given in the next section.

Geometric analysis

We first recast the dynamics in a coordinate-free manner on the state submanifold X in the form

$$\dot x = f_0(x) + 1/m \sum_{i=1}^{3} u_i f_i(x) \tag{3}$$

where x is for instance $(r, \dot r)$ (the mass equation (2) holds unchanged). Among the available coordinates for the state, an interesting choice arises from the Gauss parameters (P, e, h, L) (with $e = (e_x, e_y)$, $h = (h_x, h_y)$) since the three first ones, which describe the geometry of the osculating ellipse, are first integrals of the uncontrolled motion (the last coordinate, L, is the so-called true longitude). In (3), f_0, f_1, f_2 and f_3 belong to the Lie algebra of vector fields on X, $\mathcal{X}(X)$.

We define the acceleration $\gamma = u/m$, and study the controllability of the generated family of vector fields

$$\mathcal{F} = \{v \in \mathcal{X}(X) \mid v(x) = f_0(x) + \sum_{i=1}^{3} \gamma_i f_i(x),\ |\gamma| \le \gamma_{max}\} \tag{4}$$

where γ_{max} is a fixed positive constant. The reachable set from a given point $x \in X$ (by means of piecewise constant controls) is then equal to the orbit of the semi-group of flows of $\mathcal{F}$. Since the drift f_0 is periodic and because the tangent space at any point is spanned by the brackets of $f_0, \ldots, f_3$,

$$\mathrm{Lie}_x(\{f_0, f_1, f_2, f_3\}) = T_x X \tag{5}$$

we get the (*cf.* [10])

Proposition 1. *For any positive γ_{max}, for any points x^0, x^f in X, there exists an admissible trajectory generated by a piecewise constant acceleration γ between x^0 and x^f, $|\gamma| \le \gamma_{max}$.*

The existence of admissible trajectories for the original control problem (1-2) is then obtained by choosing u and m such that $u/m = \gamma$, which can be done using

$$m = m^0 \exp(-\beta \int_0^t |\gamma| ds) > 0,\ u = m\gamma$$

and by restricting the trajectories into a compact subset of X (it is enough to assume that r is such that $|r| \geq \rho^0$ for a given positive constant ρ^0). Since the velocity field is convex, Filippov's classical result [6] allows us to conclude to the existence of solutions for the minimum time and for the maximization of the mass problems (with fixed final time, for the latter).

As a consequence of (5) and of the involutivity of $\mathrm{Vect}(\{f_1, f_2, f_3\})$, we know that the system (4) is feedback linearizable [12]. Actually, this is obvious since (4) is equivalent to

$$\begin{aligned} \dot{r} &= v \\ \dot{v} &= \widetilde{\gamma} \end{aligned}$$

with the feedback transformation $\widetilde{\gamma} = -kr/|r|^3 + \gamma$. It is then logical to ask whether the system is state-space linearizable or not: keeping the same control γ, is it possible to find coordinates in which (4) be linear? The answer is negative, due to the fact that some of the iterated Lie brackets of $f_0, \ldots, f_3$ are not vanishing [12]:

$$(\exists i \in \{1, \ldots, 3\}) \ : \ \mathrm{ad}_{f_0} f_i \neq 0.$$

We conclude this section by an analysis of the switching structure of (4) (details for the 2D case can be found in [3]). If (x, γ) is a minimum time solution with associated transfer time t_f, Pontryagin Maximum Principle tells us that there exist a non-negative constant p_0 and an absolutely continuous Hamiltonian flow $y = (x, p)$ on the cotangent bundle T^*X such that:

(i) $(p_0, p) \neq (0, 0)$ (non-triviality)

(ii) $j_B^*(p_0 dt + H(t)dt - \sum_{i=1}^{6} p_i dx_i)(t_f, x(t_f)) = 0$ (transversality)

(iii) $H(y(t), \gamma(t)) = \min_{|v| \leq \gamma_{max}} H(y(t), v)$, a.e. on $[0, t_f]$ (minimality).

Hereabove, H is the Hamiltonian of the problem and j_B^* is the pull-back on the target set $B = \{(t, x) \in \mathbf{R} \times X \mid t > 0,\ h(t, x) = 0\}$. As a result,

$$\gamma = -\gamma_{max} \psi / |\psi|$$

whenever the switching function $\psi(t) = (H_1, H_2, H_3)(y(t))$ does not vanish ($H_i(y) = pf_i(x)$, $i = 0, \ldots, 3$). From the maximum rank condition (5) and the involutivity of $\mathrm{Vect}(\{f_1, f_2, f_3\})$ we get that the set of switching points

$$\mathcal{C} = \{t \in [0, t_f] \mid \psi(t) = 0\}$$

is finite and that, for any such point $\bar{t} \in \mathcal{C}$, $\gamma(\bar{t}+) = -\gamma(\bar{t}-)$ (angle π switching). Besides, the numerical computations show that switchings may occur at the perigee of the osculating ellipse to the trajectory. Thus, defining

$$\mathcal{C}_0 = \{t \in \mathcal{C} \mid x(t) = \text{apogee or perigee}\}$$

it turns out that

Proposition 2. *There cannot be consecutive switchings in $\mathcal{C}_0$.*

Indeed, using transversality and assuming constraint qualification, one can check that the value of the Hamiltonian at $\bar{t} \in \mathcal{C}_0$ verifies

$$\begin{aligned} H(\bar{t}) &= \alpha(\bar{t})\dot{\psi}_{i_0}(\bar{t}) \\ &= -p_0 < 0 \end{aligned}$$

for some index i_0 in $1, \ldots, 3$, where α is a non-vanishing smooth function of time. As a consequence, if $\bar{t}_1 < \bar{t}_2$ are two consecutive zeros in $\mathcal{C}_0$, one has

$$\dot{\psi}_{i_0}(\bar{t}_1)\dot{\psi}_{i_0}(\bar{t}_2) = p_0^2 > 0$$

thus contradicting the fact that the derivative of the i_0-th component of ψ has to change sign.

The analysis can be extended to the original problem (1-2) to take into account the mass variation as in [3], and one gets that there cannot be more than 3 consecutive switchings at perigee or apogee. Finally, in the mass maximization case, it is still true that there are only finitely many switchings of angle π, but one has also to consider a new kind of discontinuity in the control, namely points where u vanishes.

Minimum time transfer

Let $t_f(T_{max})$ denote the value function of (1-2) (minimum time criterion). The function is finite (existence result of previous section) and clearly decreasing. Moreover, as is proved in [5], it is right-continuous in order that it makes sense to use a discrete homotopy to obtain the values of t_f for a decreasing sequence of T_{max}. Indeed, if the current value T_{max}^c is close enough to the next one, T_{max}^+, $t_f(T_{max}^c)$ provides a good initial guess for $t_f(T_{max}^+)$.

Remark 1. The result of [5] also proves that the same is true for the maximization of the mass (with fixed final time): the value function $m^f(T_{max})$ is monotonous (increasing, since we maximize) and right-continuous. In both cases, the key property of the system is the possibility of smoothly inverting the dynamics: the control can be retrieved from the position and the mass using

$$u = m(\ddot{r} + kr/|r|^3).$$

Remark 2. It is even proved in [5] that, under stronger assumptions, the value function is continuously differentiable and that its derivative with respect to the parameter T_{max} is related to the switching function ψ by

$$dt_f/dT_{max} = -\int_0^{t_f} d/dt\ (t/m)|\psi|dt.$$

In practice, the initialization of the unknown minimum transfer time is done thanks to the heuristic constancy of $t_f(T_{max})T_{max}$, first observed in [11]. The problem is then

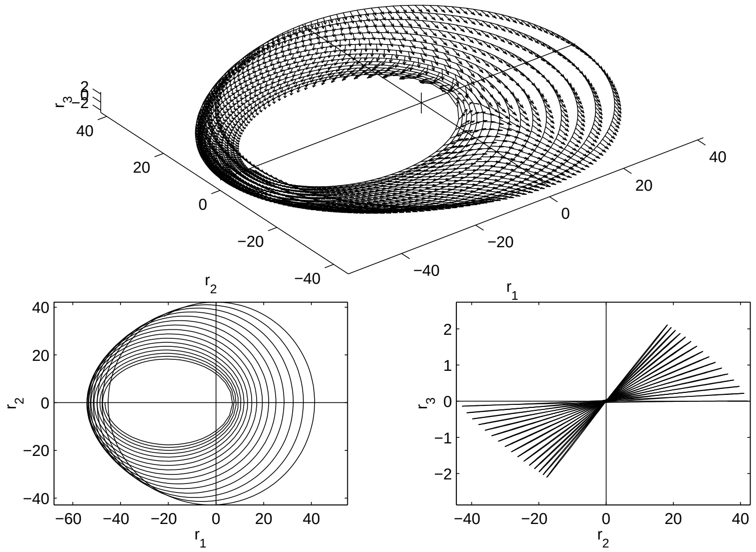

Figure 1: Minimum time transfer, T_{max} =3 Newtons ($t_f = 12$ days).

solved by single shooting on the boundary value problem obtained with Pontryagin Maximum Principle,

$$\dot{y} = \xi(t, y),\ t \in [0, t_f] \tag{6}$$

$$x(0) = x^0,\ b(y(t_f)) = 0. \tag{7}$$

If φ is the maximal flow of (6), the shooting function S is defined by

$$S(t_f, p^0) = \varphi_0^{t_f}(x^0, p^0)$$

where p^0 is the unknown initial value of the adjoint state. A discrete homotopy is used for the initialization of p^0, to go from strong thrusts to very low thrusts (around 0.1 Newton). An example of a minimum time low thrust transfer is given hereafter, Figure 1. The numerical code, `TfMin`, is freely available on request to the authors (see [4]).

Maximum mass transfer

Predictor-Corrector methods The theoretical basis can be found in [1]. The problem we consider here consists in finding a zero of a function (f) by the smooth transformation of an easier function (r). In order to do so, we introduce an homotopy map:

$$\begin{array}{rcl} H: \ R^n \times [0,1] & \to & R^n \\ (x, \lambda) & \mapsto & \left\{ \begin{array}{l} r(x)\ if\ \lambda = 0 \\ f(x)\ if\ \lambda = 1 \\ H(x,\lambda) \text{ elsewhere} \end{array} \right. \end{array}$$

The general idea of Predictor-Corrector (PC) methods is to follow the zero path of the homotopy by integrating an Initial Value Problem. Therefore, the zero path has to be differentiable (even C^2). Let us parameterize this path with the curvilinear abscissa s

and suppose that:

$$\|(\frac{\partial x}{\partial s}, \frac{\partial \lambda}{\partial s})\| = 1 \tag{8}$$

$$H(x(s), \lambda(s)) = 0 \tag{9}$$

$$H'(x(s), \lambda(s)) \ of \ maximum \ rank \ = \ n \tag{10}$$

Then after differentiation of (8) with respect to s we have:

$$[\frac{\partial H}{\partial x}(x(s), \lambda(s)), \frac{\partial H}{\partial \lambda}(x(s), \lambda(s))] \cdot \begin{bmatrix} \frac{\partial x}{\partial s} \\ \frac{\partial \lambda}{\partial s} \end{bmatrix} = 0. \tag{11}$$

Equations (6) and (9) determine (except for the direction) the unit tangent vector to the zero path. By the introduction of the augmented Jacobian matrix, we are able to compute the unique unit tangent vector to the zero path. Let us note $t(H'(x, \lambda))$ this vector at the point (x, λ). Then, following the zero path of the homotopy is equivalent to integrate the Initial Value Problem :

$$(IVP) \begin{cases} (\dot{x}(s), \dot{\lambda}(s)) = t(H'(x(s), \lambda(s))) \\ (x(0), \lambda(0)) = (x_0, 0) \end{cases}$$

The principle of PC methods is to take a step along the zero curve by predicting a point and then correcting it in order to stay on the zero curve:

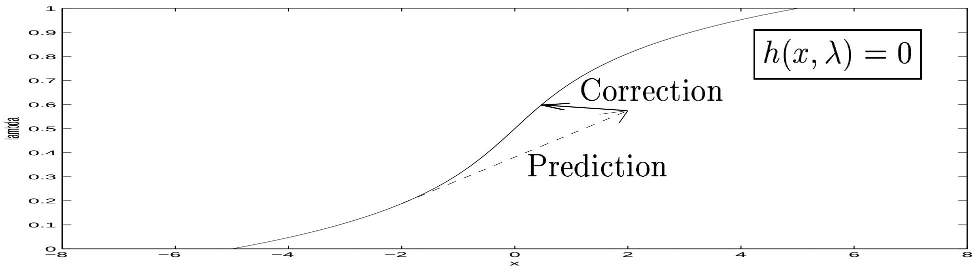

Figure 2: PC method principle.

Piecewise Linear methods Here is now a brief summary of how Piecewise Linear (PL) methods (also referred to as simplicial methods) follow the zero path of the homotopy. Readers interested in these theoretical aspects should refer to [1] and [9].

The general idea of simplicial method is to build a piecewise affine approximation of the homotopy H, and to follow the zero path of this approximation. The main advantage of this approach is that it does not require the path to be differentiable (unlike differential continuation methods for instance), continuity being enough. For this, we use a triangulation that divides the research space in simplices, as per the following definitions:

Definition 1 (Simplex and face). We call simplex the convex hull of $n+1$ affinely independants points (called the vertices) in $\mathbf{R}^n$, while a face of a simplex is the convex hull of n vertices of the simplex. For instance, in $\mathbf{R}^3$, simplices are tetrahedrons, and faces are triangles.

Definition 2 (Triangulation). A non-empty family T of simplices covering $\mathbf{R}^{n+1}$ is a triangulation of $\mathbf{R}^{n+1}$ if (i) the intersection of two simplices of T is either empty or a common face of both simplicesand (ii) the family T is locally finite (any compact subset of $\mathbf{R}^{n+1}$ meets only finitely many simplices of T).

Definition 3 (Pivoting). Let $\sigma = [v^0, .., v^{n+1}]$ be a simplex in a triangulation T of $\mathbf{R}^{n+1}$, and v^k a vertex of σ. Then there is a unique other simplex σ^* in T such that the face opposite to v^k is common to the two simplices. The passage of σ to σ^* is called a pivoting step.

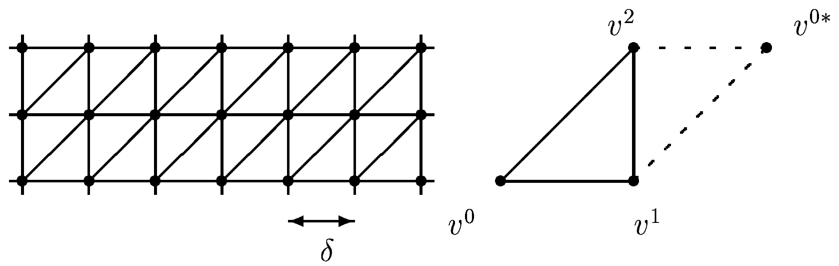

Figure 3: Simplices and pivoting step for triangulation $K_1(\delta)$ in $\mathbf{R}^2$.

Definition 4 (Labeling). We shall call labelling a map $l : simplex([v^0, .., v^{n+1}]) \mapsto \mathbf{R}^n$. In the case of the simplicial algorithm described here, the simplices will be labeled by the homotopy H: $l(v^i) = h(x^i, \lambda^i)$, where $v^i = (x^i, \lambda^i)$. Affine interpolation on the vertices thus gives a PL approximation H_T of H.

Definition 5 (Completely labeled face). A face of a simplex is completely labeled if it contains a zero of the PL approximation of the homotopy H, this property being stable under certain small perturbations.

Simplicial algorithms are based on the following fundamental property: each simplex possesses either zero or exactly two completely labeled faces. There is a constructive proof of this property, which gives the other completely labeled face of a simplex that already has a known one (this step being called the *lexicographic test*). Thus a simplicial algorithm follows the zero path of the PL approximation of the homotopy from one completely labeled face to another (via lexicographic test) within a simplex, and from one simplex to another (via pivoting step) along the zero path.

General simplicial algorithm

• *Start*
$\lambda = 0$ - First simplex with its completely labeled entry face given

• *Zero path follow-up*
While $\lambda \neq 1$ **Do**
Lexicographic test. Find the other completely labeled face ("exit face") of the current simplex.
Pivoting step. Build the other simplex sharing this face with the current simplex, which becomes the new current simplex.
Updates. Current simplex number, inverse of labeling matrix and solution.

End While

- *End*

Retrieve the coordinates of the zero of H_T when $\lambda = 1$.

Application to orbital transfer

Remark 3. The code we use for the PC method is a slight modification of the HOMPACK90 written by L.T Watson (Department of Mathematics, Michigan State University). The code for simplicial PL method has been developed in FORTRAN 90/95, and is still being updated.

Here, our target problem consists in finding the zero of a shooting function which has been obtained by application of the Maximum Principle of Pontryaguin. This problem is the orbital transfer with maximization of the final mass. Our starting problem is the orbital transfer with minimisation of the energy. So as to link those two problems, we parameterized the criterion by the homotopic parameter λ. For example the two criteria we use are:

$$Min \int_0^{t_f} (1-\lambda)\|u(t)\|^2 + \lambda\|u(t)\|dt \quad or \quad Min \int_0^{t_f} \|u(t)\|^{2-\lambda}dt$$

where u is the control (the thrust in our case).

The zero path followed from the energy criterion to the mass criterion is given in the following figures (evolution of the initial costate for the shooting method).

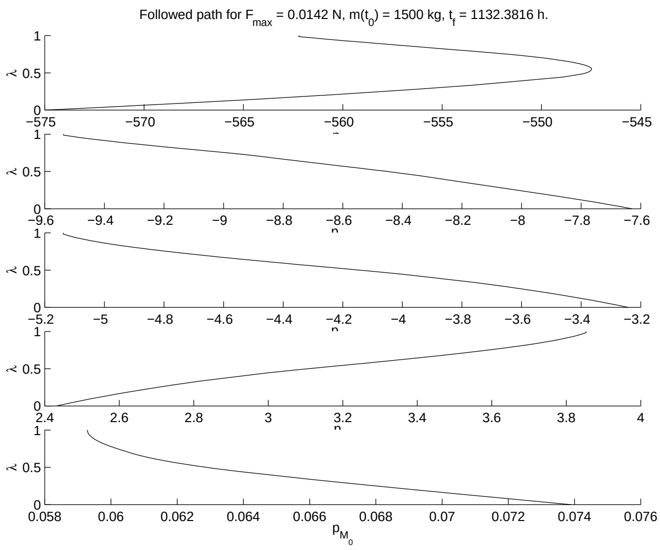

Figure 4: Zero path followed for a maximal thrust of 1 N.

Now we represent the optimal control at various stages along the zero path. The progressive deformation from the continuous optimal control corresponding to the minimization of the energy to the discontinuous solution for the maximization of the

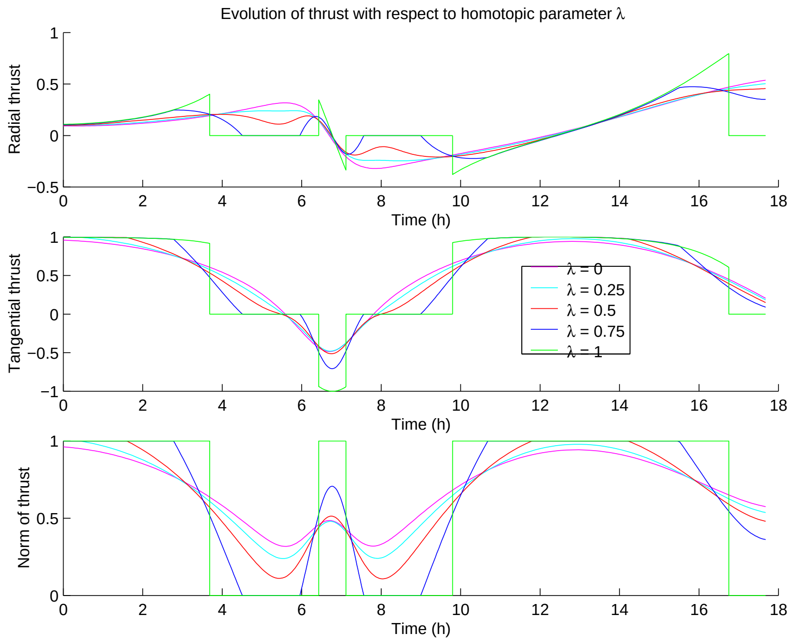

Figure 5: Optimal control with respect to homotopic parameter lambda, at 60 N.

final mass is clearly seen.

Here is the solution obtained for the mass criterion, with a maximal thrust of 1 N. The first column shows the five orbital parameters (P, e_x, e_y, L, m) (coplanar transfer), the second column shows the costate, and on the rigth are the trajectory and optimal control (radial and tangential thrusts, and norm).

The continuation method reached solutions for the mass optimization problem with maximal thrusts as low as 0.1 N, where the transfer involves about 450 revolutions and more than 900 commutations for the control.

Conclusions

If the switching structure of the minimum time controls is quite well understood, much remains to be done for the maximization of the mass problem. Besides, the same kind of geometric approach can also be used to construct analytic admissible trajectories for the system.

While the work on the maximization of the final mass is being extended to the 3D case for lower thrusts, new models that include challenging logical constraints for the minimum time transfer are also being considered in the context of hybrid control.

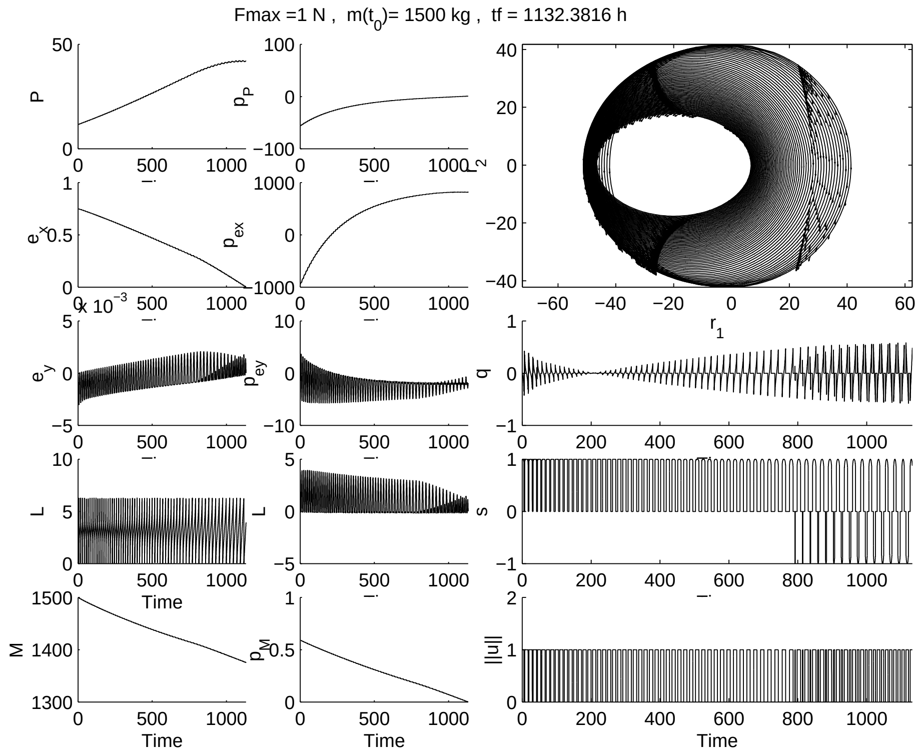

Figure 6: State, costate, trajectory and optimal control at 1 N.

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