AVERAGING AND OPTIMAL CONTROL OF ELLIPTIC KEPLERIAN ORBITS WITH LOW PROPULSION

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Abstract: This article deals with the optimal transfer of a satellite between Keplerian orbits using low propulsion and is based on preliminary results of (Geffroy, 1997) where the optimal trajectories are approximated using averaging techniques. The objective is to introduce the geometric framework and to make an analysis of the averaged optimal trajectories in the energy minimization problem, showing in particular the connection with Riemannian problems, with integrable geodesics.

Keywords: Orbital transfer, optimal control, averaging

1. INTRODUCTION

An important problem in astronautics is to transfer a satellite between elliptic orbits. Recent research projects concern orbital transfer with electro-ionic propulsion where the thrust is very low. For the sake of simplicity, we restrict ourselves to the 2D-orbital transfer, assuming the mass constant. If we decompose the thrust in the *tangential-normal frame*, the system is described by GAUSS equations :

$$\frac{de}{dt} = \frac{\sqrt{1-e^2}}{\sqrt[3]{\mu n}} \frac{1}{\sqrt{1+2e\cos v+e^2}} \left[2(e+\cos v)u_t - \sin v \frac{1-e^2}{1+e\cos v}u_n\right] \quad (1)$$

$$\frac{dn}{dt} = -\frac{3n^{\frac{2}{3}}}{\sqrt{1-e^2}\sqrt[3]{\mu}}\left[\sqrt{1+2e\cos v+e^2}\,u_t\right] \quad (2)$$

$$\frac{d\omega}{dt} = \frac{\sqrt{1-e^2}}{e\sqrt[3]{\mu n}} \frac{1}{\sqrt{1+2e\cos v+e^2}} \left[(2\sin v)u_t + \frac{2e+\cos v+e^2\cos v}{1+e\cos v}u_n\right] \quad (3)$$

$$\frac{dl}{dt} = (1+e\cos v)^2 \frac{n}{(1-e^2)^{\frac{3}{2}}} \quad (4)$$

where the coordinates are given by :

- e : eccentricity
- n : mean motion, $n = \sqrt{\dfrac{\mu}{a^3}}$, a : semi-major axis and μ is the gravitation constant
- ω : argument of the pericenter
- l : polar angle or longitude

and $v = l - \omega$ is the true anomaly, $u = (u_t, u_n)$ representing the control decomposed in the tangential-normal frame. We observe that if the thrust $|u| \le \varepsilon$ is low, the system can be written after renormalization

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as a system :

$$\dot{x} = \varepsilon \sum_{i=1}^{2} u_i F_i(x, l) \tag{5}$$

$$\dot{l} = G(x, l) \tag{6}$$

where $x = (e, n, \omega)$ are slowly varying variables, l is the fast variable and $u = (u_1, u_2)$ is the control vector, $|u| \leq 1$. Moreover F_i, G are 2π-periodic with respect to the angular variable.

In orbital transfer, we must steer the system from an initial position (x_0, l_0) to a terminal orbit represented by x_1, taking into account physical cost functions e.g. the time, the energy $\int_0^T \sum u_i^2 \, dt$ or the final mass related to $\int_0^T |u| \, dt$. Using the maximum principle the optimal trajectories are to be found among a set of extremal curves, solutions of an Hamiltonian system defined by an Hamiltonian H_ε. Such system is extremely complicated and can be analyzed using only numerical simulations. Moreover, for low propulsion we can observe on the numerical results only the averaged behaviour of the solutions. This was the starting point of Geffroy's work, making a preliminary analysis of the averaged system in the energy minimization problem (the constraint $|u| \leq 1$ being relaxed, but is satisfied at the end, adjusting the transfer time). As observed, this led to an averaged system which :

(1) can be mathematically computed using standard integrals computations,
(2) is integrable by quadrature, if we transfer the system to a geostationary orbit.

The aim of this short article is to complete the computations and to derive properties of the optimal solution. Moreover we show that *the averaged system is equivalent to a Riemannian problem* in $\mathbf{R}^3$ which can be written :

$$\frac{dx}{dt} = \sum_{i=1}^{3} u_i F_i(x) \quad , \quad \mathrm{Min} \int_0^T \sum_{i=1}^{3} u_i^2 \, dt$$

the additional control being generated by averaging the optimal control with respect to the fast variable and producing displacement in the directions generated by the Lie brackets.

2. COMPUTATIONS OF THE AVERAGED SYSTEM AND DEFINITION OF THE ASSOCIATED RIEMANNIAN PROBLEM

2.1 Averaged system

We can assume $\mu = 1$, setting $u_t = \varepsilon v_t$, $u_n = \varepsilon v_n$, where $|u| \leq \varepsilon$ and parameterizing the trajectories by the *cumulated longitude,* the energy minimization problem takes the form :

$$\frac{de}{dl} = \varepsilon D \left[2(e + \cos v) v_t - \sin v \frac{1 - e^2}{W} v_n \right] \tag{7}$$

$$\frac{dn}{dl} = -3\varepsilon \frac{1 - e^2}{\sqrt[3]{\mu}} \left[\frac{\sqrt{1 + 2e\cos v + e^2}}{W^2} v_t \right] \tag{8}$$

$$\frac{d\omega}{dl} = \varepsilon \frac{D}{e} \left[(2\sin v) v_t + \frac{2e + \cos v + e^2 \cos v}{W} v_n \right] \tag{9}$$

and the cost is

$$\int_{l_0}^{l_1} \varepsilon^2 (v_t^2 + v_n^2) \frac{(1 - e^2)^{\frac{3}{2}}}{nW^2} \, dl$$

where

$$W = 1 + e\cos v \quad , \quad D = \frac{(1 - e^2)^2}{n^{\frac{4}{3}} W^2 \sqrt{1 + 2e\cos v + e^2}}$$

and the control has to satisfy the constraint $v_t^2 + v_n^2 \leq 1$. The problem is of the form

$$\frac{dx}{dl} = \sum_{i=1}^{2} u_i F_i(x, l), \ \underset{u_1^2 + u_2^2 \leq 1}{\mathrm{Min}} \int_{l_0}^{l_1} \varepsilon^2 (u_1^2 + u_2^2) g(x, l) \, dt.$$

From the maximum principle, the associated Hamiltonian is :

$$H_\varepsilon = p_0 \varepsilon^2 (u_1^2 + u_2^2) g(x, l) + \varepsilon \sum_{i=1}^{2} u_i \langle p, F_i(x, l) \rangle$$

where p_0 can be normalized to $-\frac{1}{\varepsilon}$ and p is the adjoint vector dual to x. Relaxing the constraint $u_1^2 + u_2^2 \leq 1$, leads to compute the extremals solving $\frac{\partial H_\varepsilon}{\partial u} = 0$.

Plugging the corresponding solutions in H_ε defines the *true Hamiltonian function* :

$$\varepsilon^{-1} H_\varepsilon(x, p, l) = \frac{nW^2}{4(1 - e^2)^{\frac{3}{2}}} \Bigg[\bigg(2p_1 D(e + \cos v) - 3p_2 \frac{1 - e^2}{\sqrt[3]{n}} \frac{\sqrt{1 + 2e\cos v + e^2}}{W^2} + 2p_3 \frac{D \sin v}{e} \bigg)^2 + \bigg(-p_1 D \sin v \frac{1 - e^2}{W} + p_3 \frac{D(2e + \cos v + e^2 \cos v)}{eW} \bigg)^2 \Bigg]. \tag{10}$$

By construction H_ε is 2π-periodic with respect to the angular variable l and the mean Hamiltonian is :

$$\overline{H} = \frac{1}{2\pi} \int_0^{2\pi} \varepsilon^{-1} H_\varepsilon(x, p, l) \, dl$$

and the averaged can be computed with respect to true anomaly $v = l - \omega$.

A key observation is that the computation amounts to evaluate integrals of the form $\frac{1}{2\pi} \int_0^{2\pi} F(\cos v, \sin v)$ where F is a rational function. Therefore a tedious but straightforward computation using the residu theorem gives the following result :

Proposition 1. The averaged Hamiltonian is :

$$\overline{H} = \frac{1}{8N^{\frac{5}{3}}} \left[5(1 - E^2) Q_1^2 + 18 N^2 Q_2^2 + \frac{5 - 4E^2}{E^2} Q_3^2 \right] \tag{11}$$

where (E, N, Ω) is the notation for the variables (e, n, ω) of the averaged system and (Q_1, Q_2, Q_3) are the corresponding dual variables.

2.2 Associated Riemannian problem

We observe that $\overline{H}$ can be written as the sum of three squares : $\overline{H} = \frac{1}{2}(P_1^2 + P_2^2 + P_3^2)$ where P_1, P_2 and P_3 are the Hamiltonian lifts $P_i = \langle Q, F_i \rangle$ associated to the following vector fields :

$$F_1 = \sqrt{\frac{5}{4}(1-E^2)}\frac{1}{N^{\frac{5}{6}}}\frac{\partial}{\partial E} \quad (12)$$

$$F_2 = \sqrt{\frac{9}{2}}N^{\frac{1}{6}}\frac{\partial}{\partial N} \quad (13)$$

$$F_3 = \frac{1}{2}\frac{\sqrt{5-4E^2}}{E}\frac{\partial}{\partial \Omega} \quad (14)$$

where from the definition $N > 0$ and $E > 0$, the singularity $E = 0$ corresponding to a circular orbit. Therefore $\overline{H}$ is the Hamiltonian associated to the Riemannian problem :

$$\frac{dx}{dt} = \sum_{i=1}^{3} u_i F_i(x) \quad , \quad x = (E, N, \Omega) \quad ,$$
$$\text{Min} \int_0^T \sqrt{\sum_{i=1}^{3} u_i^2}\, dt \quad (15)$$

and according to Maupertuis principle, the length minimization is equivalent to the energy minimization :

$$\text{Min} \int_0^T \sum_{i=1}^{3} u_i^2\, dt \quad (16)$$

where $T > 0$ is fixed.

The existence of such a Riemannian equivalent problem is generalizable to the class of problems normalizable to :

$$\frac{dx}{dl} = \varepsilon \sum_{i=1}^{m} u_i F_i(x, l), x \in \mathbf{R}^n \quad ,$$
$$\text{Min}\, \varepsilon^2 \int_0^T \sum u_i^2\, dt \quad (17)$$

where the Hamiltonian is $H_\varepsilon = \dfrac{1}{2}\displaystyle\sum_{i=1}^{m} \langle p, F_i(x, l)\rangle^2$ with $H_\varepsilon \geq 0$ and hence $\overline{H} \geq 0$. Moreover if $\overline{H} > 0$, we can write $\overline{H} = \dfrac{1}{2}\displaystyle\sum_{i=1}^{n} P_i^2$, n being the dimension of the space, $P_i = \langle p, F_i \rangle$. This is in particular the case for the 3D-orbital transfer problem, see (BONNARD *et al.*, 2005*b*).

3. ANALYSIS OF THE AVERAGED SYSTEM

The equations associated to $\overline{H}$ are :

$$\frac{dE}{dt} = \frac{\partial \overline{H}}{\partial Q_1} = \frac{5}{4}Q_1 \frac{1-E^2}{N^{\frac{5}{3}}} \quad (18)$$

$$\frac{dN}{dt} = \frac{\partial \overline{H}}{\partial Q_2} = \frac{9}{2}Q_2 N^{\frac{1}{3}} \quad (19)$$

$$\frac{d\Omega}{dt} = \frac{\partial \overline{H}}{\partial Q_3} = \frac{1}{4}Q_3 \frac{5-4E^2}{E^2}\frac{1}{N^{\frac{5}{3}}} \quad (20)$$

$$\frac{dQ_1}{dt} = -\frac{\partial \overline{H}}{\partial E} = \frac{5}{4}\left(Q_1^2 E + \frac{Q_3^2}{E^3}\right)\frac{1}{N^{\frac{5}{3}}} \quad (21)$$

$$\frac{dQ_2}{dt} = -\frac{\partial \overline{H}}{\partial N} = \frac{25}{24}Q_1^2 \frac{1-E^2}{N^{\frac{8}{3}}} - \frac{3}{4}\frac{Q_2^2}{N^{\frac{2}{3}}} + \frac{5}{24}Q_3^2 \frac{5-4E^2}{E^2}\frac{1}{N^{\frac{8}{3}}} \quad (22)$$

$$\frac{dQ_3}{dt} = -\frac{\partial \overline{H}}{\partial \Omega} = 0. \quad (23)$$

Hence Ω is a cyclic coordinate and therefore $Q_3(t) = C_1$: constant. The case $Q_3 = 0$ corresponds formally to the transversality condition associated to an optimal transfer to a circular orbit. Next, it will be analyzed in details.

3.1 Averaged transfer to a circular orbit

We note G_{eo} the 4-dimensional symplectic space $\{E, N, Q_1, Q_2\}$ defined by $\Omega = Q_3 = 0$ which is invariant for the trajectories, hence we can restrict $\overline{H}$ to this space and the restriction of $\overline{H}$ is analytic and is associated to a 2D-Riemannian metric g corresponding to the system :

$$\frac{dx}{dt} = \sum_{i=1}^{2} u_i F_i(x) \quad , \quad \text{Min} \int_0^T \sum u_i^2\, dt$$

where F_1, F_2 are orthonormal vectors fields defined by :

$$F_1 = \sqrt{\frac{5}{4}}\frac{\sqrt{1-E^2}}{N^{\frac{5}{6}}}\frac{\partial}{\partial E} \quad , \quad F_2 = \sqrt{\frac{9}{2}}N^{\frac{1}{6}}\frac{\partial}{\partial N}$$

and the metric is defined by $g = g_{11}(dE)^2 + g_{22}(dN)^2$ where

$$g_{11} = \frac{4}{5}\frac{N^{\frac{5}{3}}}{1-E^2} \quad , \quad g_{22} = \frac{2}{9}\frac{1}{N^{\frac{1}{3}}}.$$

In particular g is a sum of squares and E, N are *orthogonal coordinates*. We can write g in standard coordinates.

3.1.1. Geodesic coordinates

By setting $W = \sqrt{\frac{4}{5}}\arcsin E$ and $V = \frac{2\sqrt{2}}{5}N^{\frac{5}{6}}$, the metric can be reduced to :

$$g = dV^2 + G(V)dW^2 \quad , \quad G(V) = \frac{25}{8}V^2$$

and the Gauss curvature is given by Gauss formula :

$$K = -\frac{1}{\sqrt{G}}\frac{\partial^2 \sqrt{G}}{\partial V^2} = 0.$$

Hence the GAUSS curvature of the metric is zero and the metric is locally isomorphic to the flat matrix dx^2+dy^2.

To get a global result *in the domain*, we observe that if we set

$$R = V = \frac{2\sqrt{2}}{5}N^{\frac{5}{6}} \quad , \quad \theta = \sqrt{\frac{5}{2}}\arcsin E$$

where $R > 0$, $\theta \in \left[0, \frac{\sqrt{5}}{2\sqrt{2}}\pi\right]$ the metric g takes the *polar form* $g = dR^2 + R^2 d\theta^2$ and g is globally isomorphic to dX^2+dY^2 if we set : $X = R\cos\theta, Y = R\sin\theta$.

Therefore we deduce the following important result :

Lemma 2. The GAUSS curvature of the metric is zero and in the metric is globally isomorphic to the flat metric dX^2+dY^2. Hence the geodesics are straight-lines, in suitable coordinates.

3.1.2. Liouville coordinates In order to integrate the geodesic flow, we can also observe that g is isomorphic to a Liouville metric of the form $\lambda(u,v)(du^2+dv^2)$ where $\lambda(u,v) = f(u)+g(v)$. Indeed, from the previous computations g is isomorphic to :

$$dx^2 + x^2dy^2 = x^2\left[\left(\frac{dx}{x}\right)^2 + dy^2\right] \tag{24}$$

and setting $z = \ln x$, we get $g = e^{2z}(dz^2+dy^2)$ hence $\lambda = e^{2z}$.

In those coordinates, orthonormal vector fields are defined by :

$$F_1 = \frac{1}{\sqrt{\lambda}}\frac{\partial}{\partial z} = e^{-z}\frac{\partial}{\partial z} \quad , \quad F_2 = \frac{1}{\sqrt{\lambda}}\frac{\partial}{\partial y} = e^{-z}\frac{\partial}{\partial y}$$

and the Hamiltonian takes the form :

$$H = \frac{1}{2}\left(\frac{p_z^2}{\lambda} + \frac{p_y^2}{\lambda}\right) \quad , \quad \lambda = e^{2z}.$$

In particular, y is a cyclic coordinate, p_y is a first integral defined by $\langle p, F\rangle$, where $F = \frac{\partial}{\partial y}$ and from Noether theorem the metric is invariant by F i.e. $L_F g = 0$ where L is the Lie derivative.

The integration is straightforward :

$$\frac{dz}{dt} = \frac{p_z}{\lambda} = e^{-2z}p_z \quad , \quad \frac{dy}{dt} = \frac{p_y}{\lambda} = e^{-2z}p_y$$
$$\frac{p_z}{dt} = \frac{1}{2}(p_z^2+p_y^2)\frac{1}{\lambda^2}\frac{d\lambda}{dt} = (p_z^2+p_y^2)e^{-2z}$$
$$\frac{p_y}{dt} = 0$$

and Euler-Lagrange equation is $\ddot{z} = -2e^{-2z}p_z + e^{-2z}\dot{p}_z = -2e^{-2z}p_z + e^{-4z}(p_z^2+p_y^2)$.

Hence $\ddot{z} = -2\dot{z} + (\dot{z}^2+\dot{y}^2)$. Similarly $\ddot{y} = -2\dot{y}$.

We deduce also $\ddot{p}_z = 0$, therefore $y(t) = C_1 + C_2e^{-2t}$ and $z(t)$ can be easily computed from $e^{2z}dz = p_z dt$, since $p_z(t)$ is affine.

Hence we have proved the following :

Lemma 3. The metric g is a Liouville metric with a linear first integral and the geodesic flow can be integrated using elementary functions.

The integrability property of the extremal flow restricted to G_{eo} was already observed in (Geffroy, 1997) and we shall make the integration in the original geometric coordinates (E,N), which will be extended to the full extremal system.

Indeed, restricting our equations to G_{eo} gives us :

$$\frac{dE}{dt} = \frac{\partial \overline{H}}{\partial Q_1} = \frac{5}{4}Q_1\frac{1-E^2}{N^{\frac{5}{3}}} \tag{25}$$
$$\frac{dN}{dt} = \frac{\partial \overline{H}}{\partial Q_2} = \frac{9}{2}Q_2N^{\frac{1}{3}} \tag{26}$$
$$\frac{dQ_1}{dt} = -\frac{\partial \overline{H}}{\partial E} = \frac{5}{4}Q_1^2E\frac{1}{N^{\frac{5}{3}}} \tag{27}$$
$$\frac{dQ_2}{dt} = -\frac{\partial \overline{H}}{\partial N} = \frac{25}{24}Q_1^2\frac{1-E^2}{N^{\frac{8}{3}}} - \frac{3}{4}\frac{Q_2^2}{N^{\frac{2}{3}}} \tag{28}$$

and the Hamiltonian $\overline{H}$ restricted to G_{eo} is constant :

$$\frac{1}{8N^{\frac{5}{3}}}\left[5(1-E^2)Q_1^2 + 18N^2Q_2^2\right] = K_2. \tag{29}$$

The first integral linear in Q is obtained by considering :

$$\frac{dE}{dt} = \frac{5}{4}Q_1\frac{1-E^2}{N^{\frac{5}{3}}} \quad , \quad \frac{dQ_1}{dt} = \frac{5}{4}Q_1^2\frac{E}{N^{\frac{5}{3}}} \tag{30}$$

hence

$$\frac{dE}{dQ_1} = \frac{1-E^2}{Q_1E} \tag{31}$$

and separating the variables, we get

$$\frac{d(1-E^2)}{(1-E^2)} = -2\frac{dQ_1}{Q_1} \tag{32}$$

Therefore we deduce $Q_1^2(1-E^2) = K_1'$ and the corresponding linear first integral :

$$Q_1\sqrt{1-E^2} = K_1 \tag{33}$$

associated to the isometry induced by $\sqrt{1-E^2}\frac{\partial}{\partial E}$.

Now, let $U = NQ_2$, differentiating and using $\overline{H}_{|G_{\mathrm{eo}}} = K_2$, we obtain :

$$\dot{U} = \frac{5}{3}K_2 \tag{34}$$

and setting $V = N^{\frac{5}{3}}$, we get :

$$\dot{V} = \frac{5}{3}N^{\frac{2}{3}}\dot{N} = \frac{15}{2}U \tag{35}$$

Therefore

$$\ddot{V} = \frac{25}{2}K_2. \tag{36}$$

Hence $V(t)$ is a second-order polynomial given by :

$$V(t) = \frac{25}{4}K_2t^2 + \dot{V}(0)t + V(0) \tag{37}$$

and computing the discriminant, we get :

$$\Delta = -\frac{125}{8}(1-E(0)^2)Q_1(0)^2 \tag{38}$$

which is strictly negative if $Q_1(0) \neq 0$. The integration for $Q_1(0) = 0$ is straightforward. Otherwise we have

$$\sqrt{|\Delta|} = \left(\frac{5}{2}\right)^{\frac{3}{2}} \sqrt{1 - E(0)^2} |Q_1(0)| > 0. \qquad (39)$$

To compute $E(t)$ we proceed as follows. We have :

$$\frac{dE}{dt} = \frac{5}{4}\frac{1-E^2}{N^{\frac{5}{3}}} Q_1 \qquad (40)$$

and using $Q_1\sqrt{1-E^2} = K_1$, we obtain :

$$\frac{dE}{\sqrt{1-E^2}} = \frac{5}{4} K_1 \frac{dt}{V(t)}. \qquad (41)$$

Therefore, introducing $W = \arcsin E$, we get :

$$W(t) - W(0) = \frac{5}{4} K_1 \int_0^t \frac{d\tau}{V(\tau)}. \qquad (42)$$

Denoting $V(t) = at^2 + bt + c$ and using $\Delta < 0$, we can write $V(t) = \frac{|\Delta|}{4a}\left[1 + \left(\frac{2at+b}{\sqrt{|\Delta|}}\right)^2\right]$ and setting $T = \frac{2at+b}{\sqrt{|\Delta|}}$, we obtain :

$$W(t) - W(0) = \sqrt{\frac{2}{5}} \operatorname{sign} Q_1(0) [\arctan T(t) - \arctan T(0)]. \qquad (43)$$

This gives the parameterization of the geodesic curves of the 2D-Riemannian problem underlying the transfer towards the geostationary orbit.

3.2 Integrability in the general case

We proceed as previously : Ω is a cyclic coordinate and we set $Q_3 = C_1$ constant. Introducing $U = NQ_2$, we deduce from the equations :

$$\dot{U} = \frac{1}{N^{\frac{5}{3}}} \left[\frac{25}{24}(1-E^2)Q_1^2 + \frac{15}{4}N^2 Q_2^2 + \frac{5}{24}\frac{5-4E^2}{E^2}Q_3^2\right] \qquad (44)$$

and since $\overline{H} = C_2$ constant, we have :

$$8C_2 = \frac{1}{N^{\frac{5}{3}}} \left[5(1-E^2)Q_1^2 + 18N^2 Q_2^2 + \frac{5-4E^2}{E^2}Q_3^2\right] \qquad (45)$$

which implies

$$\dot{U} = \frac{5}{3} C_2. \qquad (46)$$

If we set $V = N^{\frac{5}{3}}$, we obtain :

$$\dot{V} = \frac{5}{3} N^{\frac{2}{3}} \dot{N} = \frac{15}{2} U \qquad (47)$$

therefore

$$\ddot{V} = \frac{25}{2} C_2 \qquad (48)$$

and $V(t)$ is a polynomial of degree 2. This gives $N(t)$ and $Q_2(t)$ is deduced from the second equation.

In order to compute the remaining, we introduce :

$$\overline{H}' = 5(1-E^2)Q_1^2 + \frac{5-4E^2}{E^2}Q_3^2 \qquad (49)$$

and the corresponding Hamiltonian system in the symplectic space (E, Ω, Q_1, Q_3) will be integrated using the parameterization $dT = \frac{dt}{8N^{\frac{5}{3}}}$.

We have :

$$\frac{dE}{dT} = 10(1-E^2)Q_1 \qquad (50)$$

$$\frac{d\Omega}{dT} = 2\frac{(5-4E^2)}{E^2}Q_3 \qquad (51)$$

$$\frac{dQ_1}{dT} = 10\left(EQ_1^2 + \frac{Q_3^2}{E^3}\right) \qquad (52)$$

$$\frac{dQ_3}{dT} = 0. \qquad (53)$$

The method of integration is standard. Indeed Ω is a cyclic variable and $Q_3 = C_1$ constant, $\overline{H}'$ is an Hamiltonian function from the two symplectic variables (E, Q_1) depending upon the parameter C_1. The associated planar system is completely integrable and Ω can be computed by quadrature. More precisely the detailed computations are the following. We have :

$$\overline{H}' = 5(1-E^2)Q_1^2 + \frac{5-4E^2}{E^2}C_1^2 \qquad (54)$$

Setting $\overline{H}' = C_3^2$ and using $Q_1 = \frac{\dot{E}}{10(1-E^2)}$, we obtain :

$$(\dot{E})^2 = 20\frac{1-E^2}{E^2}\left[C_3E^2 - (5-4E^2)C_1^2\right].$$

Hence separating the variables, we have :

$$\frac{EdE}{\sqrt{1-E^2}\sqrt{P(E)}} = \pm dT \qquad (55)$$

where $P(E) = 20[(C_3^2 + 4C_1^2)E^2 - 5C_1^2]$.

We set $W = 1 - E^2$ and we get :

$$\frac{dW}{\sqrt{Q(W)}} = \pm dT \qquad (56)$$

where Q is the polynomial of degree 2 :

$$Q(W) = 80W[(C_3^2 - C_1^2) - (C_3^2 + 4C_1^2)W]. \qquad (57)$$

Therefore the integration is straightforward.

Hence in conclusion, we get the following result :

Theorem 4. The averaged system corresponding to the 2D-transfer problem is completely integrable by quadratures. Therefore the value function S solution of Hamilton-Jacobi equation can be computed and corresponds to the energy function of the associated Riemannian problem.

4. REPRESENTATION OF THE RIEMANNIAN SPHERE ASSOCIATED TO THE COPLANAR TRANSFER

Let $x = (N, E, \Omega)$, $p = (Q_1, Q_2, Q_3)$. We denote by $z(t, z_0) = (x(t, x_0, p_0), p(t, x_0, p_0))$ the extremal curve

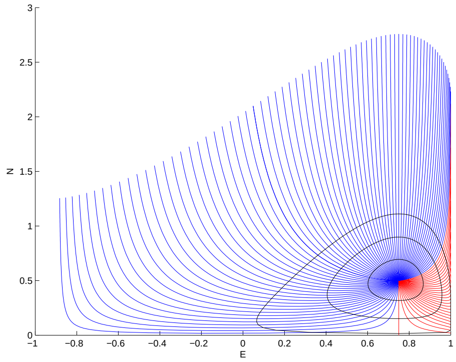

Fig. 1. Extremal curves restricted to G_{eo} up to length 1. The concentric curves are for lengths 0.1, 0.2 and 0.3

corresponding to $\overline{H}$, solution of the equations (18)-(23) with initial condition $z_0 = (x_0, p_0)$. We parameterize by arc-length $\overline{H} = \frac{1}{2}$ and if we fix x_0 the *exponential mapping* is the map $\exp_{x_0} : (p_0, t) \mapsto x(t, x_0, p_0)$. Let (p_0, t_1), $t_1 > 0$ be a point where the exponential mapping is not an immersion. Then t_1 is called a conjugate time and the image is called a *conjugate point*. The *conjugate locus* $C(x_0)$ is the set of first conjugate points. For a given extremal curve, the *cut point* is the first point where the curve ceases to be optimal and when we consider all the extremals starting from x_0, the set of such points will form the *cut locus* $L(x_0)$.

The *Riemannian sphere* with radius $r > 0$ is the set $S(x_0, r)$ of points which are at (Riemannian) distance r from x_0 and the *Riemannian ball* is the set $B(x_0, r)$ of points of distance less or equal than r from x_0. If $r > 0$ is small enough, the sphere is formed by extremities of extremal curves. Moreover the Riemannian sphere is sub-analytic if the metric is analytic and the singularities correspond to cut points which can be either a conjugate point or a point where two minimizing extremal curves with same length are intersecting. In the Riemannian case, the sphere is always smooth for r small enough, contrarily to the SR-case where conjugate points accumulate at x_0.

On figure 1, we represent extremals curves restricted to G_{eo} starting from $x_0 = (E(0), N(0)) = (0.75, 0.5)$. The line $\{E = 1\}$ corresponds to a singularity where the curves are leaving the domain of elliptic orbits. The line $\{E = 0\}$ corresponds to circular orbits and the extremal curves are prolongated by analyticity, which corresponds to a Π-singularity where the argument of the pericenter Ω rotates of π. On figure 2, we represent the sphere of radius 0.2, with $x_0 = (E(0), N(0), \Omega(0)) = (0.75, 0.5, 0)$. The inspection of figure 2 shows that the sphere is smooth, proving the global optimality of the extremal curves.

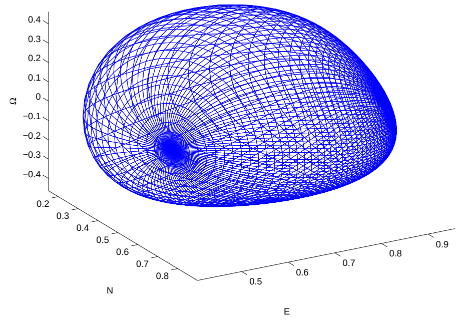

Fig. 2. Sphere of radius 0.2

5. CONCLUSION

In this short article we have investigated the coplanar transfer between Keplerian orbits, using averaging technique introduced in (Geffroy, 1997). We have proved that the extremal flow is integrable and is a geodesic flow associated to a 3D-Riemannian problem. This allows a complete solution to the averaged optimal control problem corresponding to the energy minimization problem. The complete analysis will be given in a forthcoming article. Moreover the averaged system corresponding to non coplanar transfer can be computed similarly, with no more complexity in the calculations.

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