SMOOTH HOMOTOPIES FOR SINGLE-INPUT TIME OPTIMAL ORBITAL TRANSFER

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Abstract: This article deals with the time minimal transfer of a satellite between Keplerian orbits with control along the tangential direction. The study is motivated by cone constraints on the thrust. The time optimal control law has switchings and homotopies are applied to smooth the discontinuities. The optimal solutions are computed using a shooting method, taking into account second-order optimality conditions.

Keywords: Optimal control, homotopy methods, single-input orbit transfer

1. INTRODUCTION

The orbital transfer is described by the controlled Kepler equation

$$\ddot{q} = -\mu \frac{q}{|q|^3} + \frac{u}{m} \tag{1}$$

where the mass variation is

$$\dot{m} = -\frac{|u|}{V_e} \tag{2}$$

and where q in $\mathbf{R}^3$ is the position of the satellite measured in a fixed frame whose origin is the Earth center, $\mu > 0$ is the gravitation constant, m in $\mathbf{R}_+^*$ is the mass of the satellite, u in $\mathbf{R}^3$ is the thrust of the satellite bounded in norm, $|u| = (u_1^2 + u_2^2 + u_3^2)^{1/2} \leq \varepsilon$, maximal thrust.

The system can be written

$$\dot{x} = F_0(x) + \frac{\varepsilon}{m} \sum_{i=1}^{3} u_i F_i(x) \tag{3}$$

$u = (u_1, u_2, u_3)$, $|u| \leq 1$, with the mass variation given by (2).

According to (Caillau, 2000), the time optimal control is with maximum thrust $|u| = 1$ and then $m(t)$ can be computed by (2). Moreover, except isolated singularities which can be handled numerically, an optimal control is *smooth* and is given by

$$u = \frac{(H_1, H_2, H_3)}{|(H_1, H_2, H_3)|} \tag{4}$$

where $H_i = \langle p, F_i \rangle$, $i = 1, 2, 3$, and p is the adjoint vector.

Hence time optimal controls are essentially solutions of a smooth time-dependent Hamiltonian system $\overrightarrow{H}(t, x, p)$. The same result holds for the coplanar transfer. A shooting method, taking into account second-order optimality conditions can be applied to compute numerically the optimal solution to transfer

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the satellite to the geostationary orbit (Bonnard *et al.*, 2005).

The thrust can be decomposed in the *tangential-normal frame*:

$$u = u_t F_t + u_n F_n + u_c F_c \tag{5}$$

where F_t is the unit vector colinear to $\dot q$:

$$F_t = \frac{\dot q}{|\dot q|}\frac{\partial}{\partial \dot q}. \tag{6}$$

The vector field F_c is perpendicular to the osculating plane $(q,\, \dot q)$,

$$F_c = \frac{q \wedge \dot q}{|q \wedge \dot q|}\frac{\partial}{\partial \dot q} \tag{7}$$

and $F_n = F_c \wedge F_t$.

In practice, the control may be cone-constrained (see figure 1).

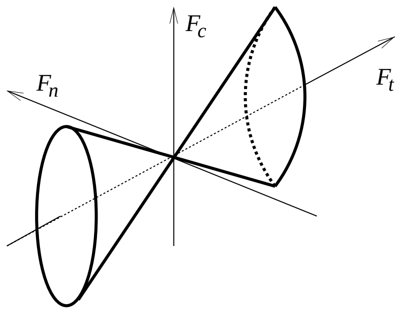

Fig. 1. Cone constraint along F_t.

Moreover, to understand the controllability properties of the system, it is important to study the effect of each control component u_t, u_n or u_c.

The objective of this article is to analyze numerically the time optimal control problem for the single-input case

$$\dot x = F_0(x) + u_t F_t(x), \quad |u| \le \varepsilon \tag{8}$$

corresponding to a transfer towards a coplanar orbit.

According to (Bonnard *et al.*, 2005), the problem restricted to the so-called elliptic domain

$$X = \{(q,\, \dot q) \,|\, q \wedge \dot q \neq 0,\ H(q,\, \dot q) < 0\}$$

with $H(q,\, \dot q) = (1/2)|\dot q|^2 - \mu/|q|$ is controllable. Besides, in the so-called *normal case* (Bonnard *et al.*, 2005), every time optimal trajectory is bang-bang i.e. $u^* = \varepsilon\, \mathrm{sign}\langle p,\, F_t\rangle$, where p is the adjoint vector, with finite number of switchings.

2. DESCRIPTION OF THE CONTINUATION METHODS

We will present continuation methods (Allgower and Georg, 1990) used at two levels. First, we make a continuation on the maximal thrust ε for the problem with multiple inputs (Caillau, 2000). Then, for a fixed ε, we set up homotopies from the multiple-input case to the single-input case. These homotopies are used to smooth the switchings which otherwise result in numerical instability.

2.1 Continuation on the maximum thrust

We consider the time optimal problem with fixed extremities for an autonomous system (everything being the same after obvious changes in the time-dependent case)

$$\dot x = f(x,\, u), \quad u \in U.$$

The associated Hamiltonian is

$$H(x,\, p,\, u) = \langle p, F(x,\, u)\rangle.$$

2.1.1. Shooting method. From Pontryagin maximum principle, every optimal trajectory is solution of the following boundary value problem:

$$\begin{aligned} &\dot x = \frac{\partial H}{\partial p}(x,p,u),\ \dot p = -\frac{\partial H}{\partial x}(x,p,u) \\ &x(0) = x_0,\ x(t_f) = x_f \end{aligned} \tag{9}$$

where u is computed using the maximization condition

$$H(x,\, p,\, u) = \operatorname*{Max}_{v \in U} H(x,\, p,\, v).$$

In the minimum time case, we define the *exponential mapping* on the $(n-1)$-dimensional projective space $\mathbf{P}^{n-1}$ as

$$\exp_{x_0,t_f}(p_0) = x(t_f, x_0, p_0) \tag{10}$$

that is as the value at t_f of the state component of the solution of the initial value problem associated with (9) and the initial condition (x_0, p_0). Solving (9) then reduces to find a zero of the *shooting function* $S\,:\, \mathbf{R} \times \mathbf{P}^{n-1} \to \mathbf{R}^n$,

$$S(t_f, p_0) = \exp_{x_0,t_f}(p_0) - x_f. \tag{11}$$

2.1.2. Continuation on the maximal thrust. The continuation on the maximal thrust (Caillau, 2000) is the generation of a sequence $(\varepsilon_n)_n$ defined as follows :

(1) We start from $\varepsilon_0 = 60$ Newtons.
(2) If ε_n is a value where the shooting method has converged i.e. if we have found a root of the shooting function, t_{f_n} and $z_n = (x_n, p_n)$ are used as initial guesses for the shooting method on an arbitrary $\varepsilon_{n+1} < \varepsilon_n$.
(3) If the shooting method has failed, we go back to step (2), with a new value of ε_n greater and closer to ε_{n-1}.
(4) We continue the process until we reach the desired ε.

Repeating the argument of (Caillau *et al.*, 2003), we have

Proposition 1. Assuming the admissible trajectories stay in a fix compact, the minimum time is a right-continuous function of the bound on the thrust.

It is thus relevant to use a decreasing sequence of thrusts to get an approximation of the limit value function.

Remark 2. In Kepler case, the compactness assumption of Proposition 1 means that there is no minimizing sequence of trajectories coming arbitrarily close to the boundary of the elliptic domain, that is arbitrarily close to a collision, parabolic trajectories, or $|q| = +\infty$.

2.2 Smooth homotopies

The idea of smoothing switchings has already been developed for the minimum consumption problem using differential homotopy methods (Gergaud and Haberkorn, 2006, to appear).

2.2.1. General algorithm. The main argument is to build an homotopy $(P_\lambda)_{\lambda\in[0,\,1]}$ from (P_0) to the single-input orbital transfer (P_1) such that (P_λ) is smooth for $\lambda \in [0,\,1[$. We require that the starting problem (P_0) can be solved for every ε using the continuation described above.

With this assumption, we proceed as follows. Given an initial step λ^0_{step} and a target λ^*,

1. We start from $\lambda_1 = 0$ and $\lambda_{\text{step}} = \lambda^0_{\text{step}}$.
2. If λ_n is such that the shooting method on (P_λ) converges, then $\lambda_{n+1} = \lambda_n + \lambda_{\text{step}}$.
3. Otherwise, λ_n is changed into $\lambda_{n-1}+\alpha\lambda_{\text{step}}$ with $0 < \alpha < 1$ and λ_{step} is changed into $\alpha\lambda_{\text{step}}$.
4. We stop either if λ^* is reached or if λ_{step} becomes smaller than a given $\lambda^{\min}_{\text{step}}$ (that is if the progression on the homotopy path is not significant enough).

The values used in the simulations are the following: $\lambda^0_{\text{step}} = 1e-1,\, 1e-2,\, 1e-3$, $\lambda^* = 9.99e-1$, $\alpha = 1e-1$, $\lambda^{\min}_{\text{step}} = 1e-7$.

We consider two different initial smooth problems (P_0), hence two different kinds of homotopy. We recall that these homotopies are performed for a given value of the maximal thrust ε.

2.2.2. Continuation on the control domain. We consider (P_0) as the transfer to a coplanar orbit by setting $u_c = 0$, i.e. $u = u_t F_t + u_n F_n$.

The only difference between (P_0) and (P_1) is the set of admissible controls: for (P_0), U_0 is the disc of centre $0_{\mathbf{R}^2}$ and radius ε, whereas for (P_1), U_1 is the segment line $[-\varepsilon, \varepsilon]$ directed along the tangential direction. The homotopy can therefore be defined as follows. The problem (P_λ) is the orbital transfer with control domain U_λ where U_λ is the ellipse of centre $0_{\mathbf{R}^2}$, semi-major axis ε along the tangential direction, and semi-minor axis $(1-\lambda)\varepsilon$ along the normal direction. Except at isolated singularities (Caillau, 2000), the problem (P_λ) is smooth for $\lambda \in [0,\,1[$, and associated with the true Hamiltonian function

$$H_\lambda(t,x,p) = H_0 + \frac{\varepsilon}{m(t)}\left[H_1^2 + (1-\lambda)H_2^2\right]^{1/2} \tag{12}$$

with, as before, $H_i = \langle p, F_i\rangle$, $i = 0, 3$, and $H_\lambda \to H = H_0 + (\varepsilon/m)|H_1|$ when $\lambda \to 1$.

If we define on $[0,1]$ the value function $\lambda \mapsto t_f(\lambda)$, the same argument as in Proposition 1 results in

Proposition 3. If the admissible trajectories remain in a fix compact independent of λ, then the value function is continuous at $\lambda = 1$.

Sketch of the proof. The result proceeds from the fact that the sequence of control sets $(U_\lambda)_\lambda$ is decreasing. Let $(\lambda_k)_k$ converge to 1 in $[0,1]$. The compactness assumption and the convexity of the dynamics ensure existence of a solution (t_{f_k}, x_k, u_k) for every k, and the sequence $(t_{f_k})_k$ is bounded over by the value of (P_1), $t_f(1)$. Taking a subsequence, we can assume it converges towards some $T \le t_f(1)$. We can also assume that the equicontinuous bounded family $(x_k)_k$ converges uniformly to some x. Now, $\dot{x}_k \in f(t,x_k,U_{\lambda_k}) \subset f(t,x_k,U_0)$ almost everywhere for all k, $f(t,x,u)$ being the dynamics, and $f(t,x,U_0)$ is convex so that $\dot{x} \in f(t,x,U_0)$ [see, *e.g.*, (Lee and Markus, 1986), theorem 4 of chap. 4]. Since U_0 is compact, we can select a measurable control u such that $\dot{x} = f(t,x,u)$. Clearly, the u_k must converge to u for the weak dual space topology of L^∞, and the uniform boundedness principle tells us that

$$\|u\|_\infty \le \liminf_k \|u_k\|.$$

In particular, the same holds for each component of the control and, since the normal controls verify $\|u_{n,k}\|_\infty \le (1-\lambda_k)\varepsilon$, we get $u_n = 0$ and $\|u_t\|_\infty \le \varepsilon$ on the limit. Then (T,x,u) is admissible and necessarily optimal since $T \le t_f(1)$ implies $T = t_f(1)$. We have shown that $(t_f(\lambda_k))_k$ converges to $t_f(1)$. $\square$

Remark 4. Compactness results on the adjoint variable similar to those on the state and the control in the above proof can also be obtained in the *normal case*.

2.2.3. Continuation on the inclination. We impose that, in contrast with the final one, the initial orbit does not belong to the equatorial plane, and we make a convex homotopy on the initial inclination (that is on the initial condition):

$$h_{x,0}(\lambda) = (1-\lambda)\eta \tag{13}$$

with $\eta \neq 0$. Indeed, the single-input transfer is a coplanar transfer and we use the following result (Caillau, 2000).

Lemma 5. Every extremal trajectory for the coplanar orbit transfer problem is also extremal for the general orbit transfer problem, provided the initial and final inclinations are the same.

We define (P_0) by setting $u_n = 0$, i.e. $u = u_t F_t + u_c F_c$, and connect so the single-input transfer to a problem with two controls, including a non-coplanar thrust.

Remark 6. Though the initial state stays in a fix compact, no such result as Proposition 3 is available for this new homotopy since there is no obvious monotonicity property on the associated value function.

3. CONJUGATE POINTS

We present briefly the concept of conjugate point in the minimum time case. We refer the reader to (Bonnard *et al.*, to appear) for details.

3.1 Definitions and properties

We consider from a *smooth* Hamiltonian system denoted $\dot{z} = \overrightarrow{H}(z)$ with $z = (x, p)$, and define the Jacobi equation which is the variational equation $\delta\dot{z} = d\overrightarrow{H}(z)\delta z$. The non-trivial solutions of this equation are called *Jacobi* fields.

Definition 7. Let $J = (\delta x, \delta p)$ be a Jacobi field, J is said to be *vertical* at time t if $\delta x(t) = 0$.

Definition 8. A time $t_c > 0$ is said to be *conjugate* if there exists a Jacobi field vertical at time $t = 0$ and $t = t_c$. Then $x(t_c)$ is called a *conjugate point*.

According to the definition (10) of exponential function in the minimum time case, we have

Proposition 9. A time t_c is conjugate if and only if the exponential mapping at t_c is not an immersion.

The key result of this theory is stated below (Sarychev, 1982).

Theorem 10. Every extremal is locally $\mathscr{C}^1$-optimal up to the first conjugate time.

In particular, the two smoothing homotopies of the previous section are obviously such that the result hereafter holds.

Proposition 11. Provided there is no conjugate point, the mapping $\lambda \mapsto p_0(\lambda)$ associated with the homotopy 2.2.2 (resp. 2.2.3) where $p_0(\lambda)$ is the zero of the corresponding shooting function, is smooth for $\lambda < 1$.

Proof. Let us consider for instance the first homotopy. For λ in $[0, 1[$, the initial adjoint state value $p_0(\lambda)$ is a zero of the shooting function

Variable	Value	
ε	6	Newtons
μ	5165.8620912	$\mathrm{Mm}^3.\mathrm{h}^{-2}$
V_e^{-1}	0.0142	$\mathrm{Mm}^{-1}.\mathrm{h}$

Table 1. Physical values.

$$\begin{aligned} S(t_f, p_0, \lambda) &= \exp_{x_0, t_f, \lambda}(p_0) - x_f \\ &= x(t_f, x_0, p_0, \lambda) - x_f \end{aligned}$$

associated with the smooth Hamiltonian (12). The assumption of non-conjugacy along the extremal ensures that the implicit function $\lambda \mapsto p_0(\lambda)$ is well defined and smooth (see Proposition 9, adding the fact that the derivative of the exponential mapping is the dynamics which defines a direction not included in the span of $\partial S/\partial p$). The same holds for the second homotopy since, according to (13), $\lambda \mapsto x_0(\lambda)$ is smooth. $\square$

3.2 Computation of conjugate points

Let $J_i = (\delta x_i, \delta p_i)$, $i = 1, \ldots, n$ be a basis of the space of Jacobi fields which are vertical at time $t = 0$. There exists a non-trivial Jacobi field which is vertical at $t = 0$ and $t = t_c$ if and only if the rank of the matrix $C(t)$ whose columns are $(\delta x_j(t))_{1 \leq j \leq n}$ is strictly less than $n - 1$ at $t = t_c$.

Our test about the rank is provided by a singular value decomposition on the matrix $C(t)$. If $\sigma_{n-1}(t)$ is the smallest singular value, the test is $\sigma_{n-1}(t_c) = 0$.

4. NUMERICAL COMPUTATIONS

For numerical reasons, we choose equinoctial coordinates (Caillau, 2000) $(P, e_x, e_y, h_x, h_y, L)$ where P is the semi-latus rectum, $e = (e_x, e_y)$ the eccentricity vector, $h = (h_x, h_y)$ the inclination vector, and L the longitude. The first five coordinates are *slow variables* corresponding to the first integrals of the free motion, while L is the *fast variable*. The numerical values used for the computation are summarized in tables 1 and 2.

Remark 12. The final longitude is actually free and the shooting function definition (11) has to be readily modified in order to take into account the transversality condition $p_L(t_f) = 0$. Practically, having obtained the relevant extremal, we compute another extremal, close to the previous one but with fixed L_f so as to be in the standard framework of conjugate point of curves with fixed extremities.

4.1 Evolution of the optimal control along homotopies

We present in figures 2 and 3 the evolution of the optimal thrust along the homotopy path respectively

	Initial conditions		Final conditions	
P	11.625	Mm	42.165	Mm
e_x	0.75		0	
e_y	0		0	
h_x	0.0612	rad	0	rad
h_y	0	rad	0	rad
L	π	rad	free	
m	1500	kg	free	

Table 2. Boundary conditions.

for the homotopy on the control domain and the homotopy on the inclination.

We can see that switchings are localized at the very beginning of the homotopy path. The remaining part of the homotopy path confirms this localization and tends to give the final shape of the optimal control. This phenomenon has already been observed in (Gergaud and Haberkorn, 2006, to appear) for the minimum consumption problem where the homotopy consists in deforming an L^2-cost into an L^1-cost.

As a first comparison, we can also remark that the localization is far more efficient in the case of the homotopy on the inclination.

4.2 *Computation of conjugate times*

Using the `cotcot` algorithm sketched above and the underlying software (Bonnard *et al.*, to appear), we can apply the conjugate points test on the intermediate problems (P_λ) for $\lambda \in [0, 1[$, since they give smooth controls.

Once we have obtained an extremal by the shooting method, we extend this extremal up to several times the minimum time. Then we apply our test to the extended extremal. We shall notice that it is a condition of optimality in the case of *fixed extremities*.

We present in the figures 4 and 5 the evolution of the smallest singular value along the homotopy path respectively for the homotopy on the control domain and the homotopy on the inclination.

We can notice that we have conjugate times at roughly three times the final time obtained by the previous shooting method, which confirms previous results (Bonnard *et al.*, 2005).

4.3 *Analysis of the extremal trajectories*

We observe that the zone where $u = \varepsilon$ (acceleration phase) is located around the apocenter. In contrast, the zone where $u = -\varepsilon$ (deceleration phase) is located around the pericenter. The apocenter is indeed the point where the gravitation is the weakest. Therefore, it is the place where the acceleration is the most efficient. Conversely, the deceleration is the most efficient when the gravitation is the strongest, that is at the pericenter. Finally, a preliminary interesting constatation on single-input transfers is that, compared to coplanar transfers with two thrusters, the minimum time is only increased of approximatively twenty percent. As illustrated by the second homotopy, a similar approach with two thrusters instead of three can be considered for non-coplanar transfers.

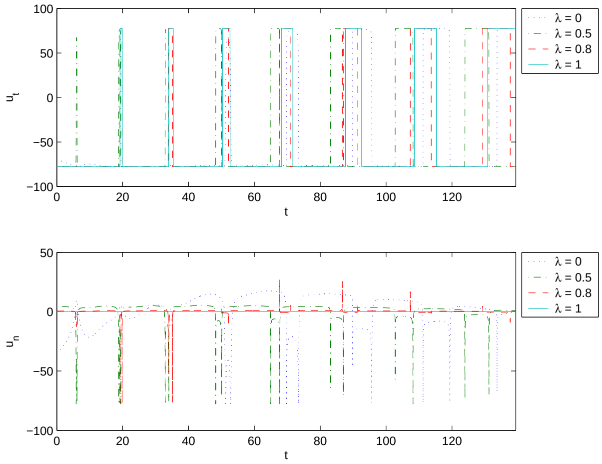

Fig. 2. Optimal control: Homotopy on the control domain.

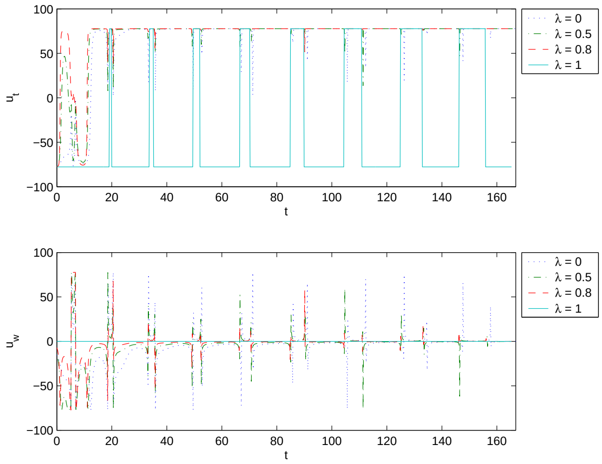

Fig. 3. Optimal control: Homotopy on the inclination.

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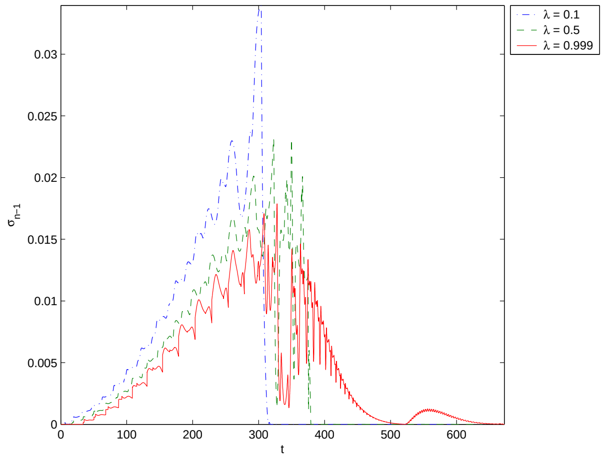

Fig. 4. Smallest singular value: Homotopy on the control domain.

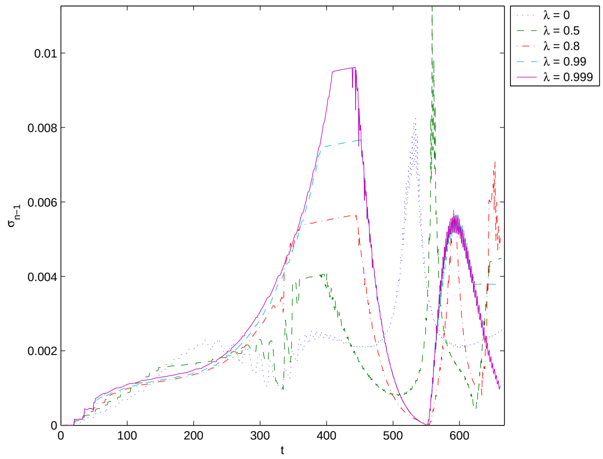

Fig. 5. Smallest singular value: Homotopy on the inclination.