On the injectivity and nonfocal domains of the ellipsoid of revolution

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Abstract In relation with regularity properties of the transport map in optimal transportation on Riemannian manifolds, convexity of injectivity and nonfocal domains is investigated on the ellipsoid of revolution. Building upon previous results [4, 5], both the oblate and prolate cases are addressed. Preliminary numerical estimates are given in the prolate situation.

Introduction

It is known after the work of Brenier [7] and McCann [12] that, under suitable assumptions, the optimal transport map between two probability measures on a compact Riemannian manifold X exists and is unique when the cost is the square of the geodesic distance, d. The issue of the continuity of this map is addressed in a series of papers of Figalli *et al.* (*cf.* [9, 10] and references therein). A crucial object in this respect is the Ma-Trudinger-Wang tensor,

$$\sigma_{(x,v)}(\xi,\eta) := -\frac{3}{2}\,\frac{\mathrm{d}^2}{\mathrm{d}s^2}\bigg|_{s=0}\,\frac{\mathrm{d}^2}{\mathrm{d}t^2}\bigg|_{t=0}\,\frac{1}{2}d^2(\exp_x(t\xi),\exp_x(v+s\eta))$$

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defined at $x \in X$, $v \in I(x)$ and $(\xi, \eta) \in T_x X \times T_x X$, where $I(x) \subset T_x X$ denotes the injectivity domain of x (see §1). On surfaces, positivity of this tensor, namely

$$\xi \perp \eta = 0 \implies \sigma_{(x,v)}(\xi, \eta) \geq 0, \quad (x, v) \in TX, \quad v \in I(x), \quad (\xi, \eta) \in T_x X \times T_x X,$$

together with convexity of the injectivity domain $I(x)$ for all points x are proved to be necessary and sufficient for the continuity of the optimal transport map. (A gap exists in dimension greater than two [10].) Using the fact that the exponential mapping is a local diffeomorphism prior to the first conjugate time, the tensor can be extended to the nonfocal domain of x, $\mathrm{NF}(x) \supset I(x)$ (see §1), and similar results involving the convexity of the nonfocal domain can also be formulated: On surfaces, positivity of the extended tensor on nonfocal domains together with convexity of all these domains are sufficient for the continuity of the optimal transport map. The ellipsoid of revolution provides a one-parameter example whose geometry is rich enough to illustrate the change in convexity of the two types of domains, injectivity and nonfocal. It has been considered in the oblate case (ellipsoid squeezed along its axis of revolution) in [4,5]. As a deformation of the round sphere, it paves the way for a systematic study of surfaces of revolution whose integrable geodesic flow has a prescribed transcendency. On the ellipsoid of revolution, the quadratures are parameterized by a complex curve of genus one, and only elliptic functions (and primitives) are required (see also [6] for the general ellipsoid).

The paper is organized as follows: In Sect. 1, the main definitions are recalled; a unified framework using a parameterization by an elliptic curve is provided, which lays the emphasis on the role of singularities of this curve to understand convexity properties of the domains. It is moreover important to use a Hamiltonian point of view that allows to interpretate the limit case of the oblate ellipsoid flattened onto a two-sided disk in connection with almost-Riemannian metrics [1,3]. Sects. 2 and 3 are devoted to the oblate and prolate cases, respectively. It is proven that the nonfocal domain of a point on the equator is not convex for an oblate enough ellipsoid. In the prolate case, numerical estimates of the curvature are given using a suitable compactification suggesting that, for a sufficiently large semi-major axis, convexity holds for injectivity domain, not for nonfocal ones.

1 Preliminaries

For $\mu > 0$, consider the ellipsoid of revolution with z-axis embedded in $\mathbf{R}^3$, $x^2 + y^2 + z^2/\mu^2 = 1$. For $\mu < 1$ (*resp.* $\mu > 1$), one has an *oblate* (*resp.* prolate) ellipsoid, while for $\mu = 1$ the round sphere is retrieved. For $(\theta, \varphi) \in \mathbf{R} \times (0, \pi)$,

$$x = \sin\varphi \cos\theta, \quad y = \sin\varphi \sin\theta, \quad z = \mu \cos\varphi,$$

is the universal covering of the ellipsoid minus its poles. In the associated coordinates (θ, φ), the metric reads $X\mathrm{d}\theta^2 + (1 - X/\lambda)\mathrm{d}\varphi^2$ with $X := \sin^2\varphi$ and $\lambda := 1/(1-\mu^2)$. We set $\lambda = \infty$ when $\mu = 1$ (round sphere), and use indifferently μ or λ to specify the geometry of the surface in the sequel. From the Hamiltonian point of view, one sets

$$H(\theta,\varphi,p_\theta,p_\varphi) := \frac{1}{2}\left(\frac{p_\theta^2}{X}+\frac{p_\varphi^2}{1-X/\lambda}\right).$$

Because of the symmetry of revolution, θ is a cyclic variable so p_θ is a linear first integral (Clairaut constant); the geodesic flow is integrable and arc length geodesics are Hamiltonian integral curves on $\{H=1/2\}$.

Proposition 1. *The quadrature on φ is parameterized by the complex curve*

$$Y^2 = 4(X-p_\theta^2)(X-1)(X-\lambda),\quad X=\sin^2\varphi,\quad Y=\frac{\dot X(\lambda-X)}{\sqrt{\lambda}},$$

which is elliptic outside singularities.

Proof. On $\{H=1/2\}$, $p_\varphi^2/(1-X/\lambda) = 1-p_\theta^2/X$ and one has

$$\dot\varphi = \frac{\partial H}{\partial p_\varphi} = \frac{p_\varphi}{1-X/\lambda}.$$

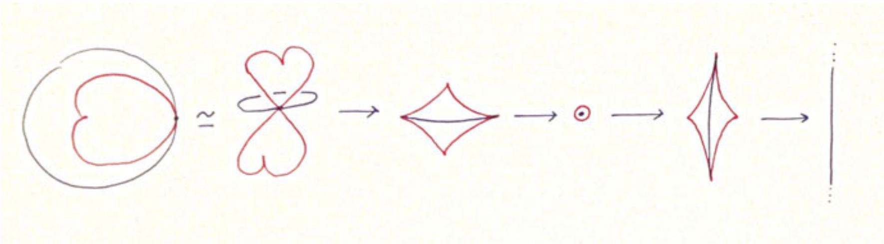
Since $\dot X^2 = 4X(1-X)\dot\varphi^2$, the result follows. □

When $\mu<1$ (oblate ellipsoid), λ is positive and the real cubic ($Y\in\mathbf{R}$) has to be used; on the converse, when $\mu>1$ (prolate ellipsoid), λ is negative and the parameterization is obtained considering the imaginary cubic ($Y\in i\mathbf{R}$). In both cases, as $p_\theta^2\le X=\sin^2\varphi\le 1$, the bounded component of the cubic is used. The complex curve is homeomorphic to some torus $\mathbf{C}/\Lambda$ where $\Lambda=\omega\mathbf{Z}+\omega'\mathbf{Z}$ is the real-rectangular lattice of periods. In the oblate (*resp.* prolate) case, X is ω-periodic (*resp.* ω'-periodic) as a function on the torus. (The period of φ is twice the period of $X=\sin^2\varphi$, and the period as a function of time is given by some time law).

The singularities are the following. When $\mu=0$, $\lambda=1$ and the elliptic curve degenerates to a rational one; geometrically, the ellipsoid is flat and the resulting singular metric is simply the flat metric on a two-sided disk (see Proposition 3). When $\mu=1$, $\lambda=\infty$ and the curve also degenerates for all p_θ; one has the round sphere whose geodesics are indeed rational curves. For any μ, $p_\theta=\pm1$ (allowed only when $X=1$) corresponds to the equator and is also a degeneracy of the elliptic curve. Finally, when $\mu=\infty$, $\lambda=0$ and the curve degenerates for $p_\theta=0$ (meridians); one may expect to use this, together with some compactification, to establish convexity properties in the prolate case, μ big enough (see the preliminary discussion §3). The bifurcations occuring in the cut and conjugate loci when going from $\mu=0$ to $\mu=1$, then to $\mu=\infty$ are portrayed Fig. 1. (See Sects. 2 and 3 on the structure of these sets in the oblate and prolate settings).

Given an initial point x_0 on the ellipsoid, consider the geodesic γ defined by $p_0\in H^{-1}(x_0,\cdot)(\{1/2\})$; as the manifold is compact,

$$t_{\mathrm{cut}}(x_0,p_0,\mu) := \sup\{t>0 \mid \gamma \text{ is minimizing on } [0,t]\}$$



$\mu = 0$ $\mu \in (0, 1)$ $\mu = 1$ $\mu \in (1, \infty)$ $\mu = \infty$

Fig. 1 Bifurcation of the cut and conjugate loci of $\varphi_0 = \pi/2$ when μ goes from $\mathbf{0}$ to $\mathbf{1}$, then to ∞. See §2 for the interpretation when $\mu = \mathbf{0}$. When $\mu = \infty$, the cut locus is the vertical line antipodal on the cylinder to the initial point (not a pole), and the conjugate locus is empty (see §3)

is finite, and is called the cut time along γ. As a subspace of the cotangent space at x_0, the injectivity domain of x_0 is defined according to

$$I(x_0) := \{t_{\rm cut}(x_0, p_0, \mu) p_0 \mid H(x_0, p_0) = 1/2\}.$$

As convexity is invariant by linear transformations, whether the injectivity domain is defined as a subspace of the tangent or cotangent fibre does not matter. The exponential mapping is

$$\exp_{x_0}(t, p_0) := x(t, x_0, p_0), \quad (t, p_0) \in \mathbf{R} \times H^{-1}(x_0, \cdot)(\{1/2\}),$$

where $(x(., x_0, p_0), p(., x_0, p_0))$ is the integral curve of H for initial condition (x_0, p_0) (globally defined on the compact manifold). Along γ, the time t is said to be conjugate if (t, p_0) is a critical point of $\exp_{x_0}$; the first of such times, if any, is called the (first) conjugate time along γ and is denoted $t_c(x_0, p_0, \mu)$. The corresponding critical value is the (first) conjugate point. One defines the nonfocal domain of x_0 as

$$\mathrm{NF}(x_0) := \{t_c(x_0, p_0, \mu) p_0 \mid H(x_0, p_0) = 1/2\}.$$

Up to the dilation $(x, y) \mapsto (x/\sqrt{X_0}, y/\sqrt{1 - X_0/\lambda})$ which does not change convexity, the boundary of $I(x_0)$ is parameterized by

$$\mathbf{S}^1 \ni \alpha \to t_{\rm cut}(x_0, p_0, \mu) \exp(i\alpha), \quad \alpha = \arg\left(\frac{p_\theta}{\sqrt{X_0}} + i\frac{p_{\varphi 0}}{\sqrt{1 - X_0/\lambda}}\right). \tag{1}$$

One can also parameterize by $p_\theta = \cos\alpha\sqrt{X_0}$, allowing a ramification above $p_{\varphi 0} = 0$ since

$$p_{\varphi 0} = \pm\sqrt{1 - X_0/\lambda}\,\sqrt{1 - p_\theta^2/X_0}.$$

(Two distinct geodesics are generated depending on the sign.) For the sake of simplicity, we denote $\tau(p_\theta) := t_{\rm cut}(x_0, p_0, \mu)$ and $' := \mathrm{d}/\mathrm{d}p_\theta$. Convexity of the injectivity domain holds if and only if the curvature of its boundary (provided the boundary is regular enough) is nonnegative.

Proposition 2. *The curvature of the injectivity domain of x_0 is*

$$K = X_0^{3/2} \frac{\tau(\tau + p_\theta \tau') + (X_0 - p_\theta^2)(2\tau'^2 - \tau\tau'')}{[(X_0 - p_\theta^2)(\tau + p_\theta\tau')^2 + (p_\theta\tau - (X_0 - p_\theta^2)\tau')^2]^{3/2}}, \quad p_\theta^2 \leq X_0,$$

whose sign is given by

$$\kappa := \tau(\tau + p_\theta\tau') + (X_0 - p_\theta^2)(2\tau'^2 - \tau\tau'').$$

Proof. In cartesian coordinates, $K = (x''y' - x'y'')/(x'^2 + y'^2)^{3/2}$ whenever defined, hence the result. □

2 Oblate case

Let $x_0 = (\theta_0, \varphi_0)$; thanks to the symmetry of revolution, one can set $\theta_0 = 0$. The initial condition is thus reduced to φ_0, that is to $X_0 = \sin^2 \varphi_0$.

Lemma 1. *The cut time along a geodesic (not a meridian) is equal to the half-period of the φ-coordinate. As such, $\tau = \tau(p_\theta, \mu)$ is independent of X_0, and of the sign of $p_{\varphi 0}$ (no ramification[1]). The injectivity domain has two axial symmetries, and convexity can be checked on a quarter of the domain.*

Proof. When $\mu < 1$, cut points are generated by the discrete symmetry $p_{\varphi 0} \mapsto -p_{\varphi 0}$: the associated geodesics intersect at $t = T/2$ where T is the period of φ. The period does not depend on the initial condition since, up to a translation on θ, any geodesic can be seen as a geodesic with initial condition $\varphi_0 = \pi/2$. The limit case $p_{\varphi 0} = 0$ (where the cut point is a conjugate point) is obtained letting p_θ tend to $\pm\sqrt{X_0}$. Because of the symmetry involved, $\tau(p_\theta, -p_{\varphi 0}, \mu) = \tau(p_\theta, p_{\varphi 0}, \mu)$ and one has an x-axis symmetry on the injectivity domain. Obviously, $p_\theta \mapsto -p_\theta$ induces another symmetry (wrt. y-axis) on the domain. □

When $\mu = 0$ ($\lambda = 1$), the metric is singular at $X = 1$ (that is $\varphi = \pi/2$). Setting $\rho = \sin\varphi$, one gets

$$X\mathrm{d}\theta^2 + (1 - X/\lambda)\mathrm{d}\varphi^2 = \sin^2\varphi\mathrm{d}\theta^2 + \mathrm{d}\rho^2 = \mathrm{d}x^2 + \mathrm{d}y^2$$

which is the flat metric on the Poincaré disk $\mathbf{D}$. Geometrically, the ellipsoid is collapsed on the unit disk and the equatorial singularity corresponds to the boundary. Crossing $\partial\mathbf{D}$ is interpretated as going from one side of the disk to the other, that is crossing the equator to go from one hemisphere to the other on the flat ellipsoid. As

[1] This is not true anymore for conjugate times outside polar or equatorial points; only one axial symmetry is preserved, see Fig. 4.

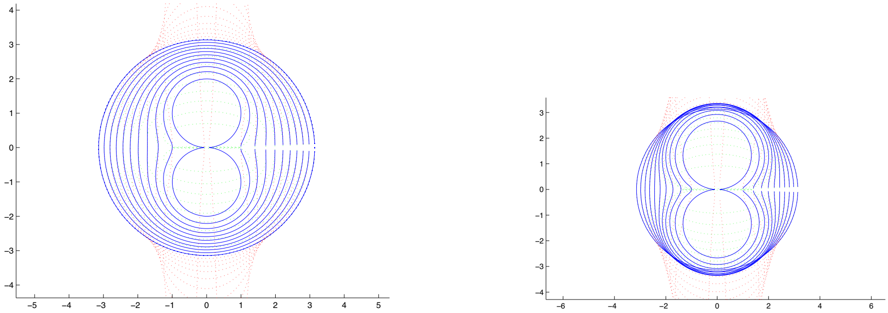


Fig. 2 Injectivity and nonfocal domains (left and right, respectively) of $\varphi_0 = \pi/2$ in the oblate case when $\mu \to 0$. For $\mu = 1$, both domains are disks, while for $\mu = 0$ both are union of tangent disks (of radii 1 and $4/3$, respectively)

the metric is flat, geodesics are straight lines, in accordance with the degeneracy of the parameterization by the elliptic curve in Proposition 1 (double root $X = 1$ when $\lambda = 1$), which trivializes the computation of the cut locus.

Proposition 3. *For $\mu = 0$ and $\varphi_0 = \pi/2$, the cut locus is the equator minus the initial point. The injectivity domain is the union of two unit disks both tangent to the x-axis at the origin, and is not convex (Fig. 2).*

Proof. The geodesic from any point on the boundary is a straight line segment which meets again the boundary; the resulting point is a cut point as is intersects the geodesic from the other side of the disk, and the cut time is just given by the common length of these segments. The whole boundary but the initial the point is so made of cut points. In parameterization (1), $\tau(\alpha) = 2\sin\alpha$ and $\alpha \mapsto \pm\tau(\alpha)\exp(i\alpha)$, $\alpha \in (0, \pi)$, is the union of two circles tangent at the origin and of radii one. $\square$

Remark 1. When $\mu = 0$ and $X_0 = 1$,

$$p_{\varphi_0} = \pm\sqrt{1 - X_0}\,\sqrt{1 - p_\theta^2/X_0} = 0,$$

so the dilation used in (1) desingularizes the parameterization of the boundary of the injectivity domain (which would otherwise collapse on a segment), revealing its non-convexity.

By continuity, $I(\varphi_0 = \pi/2)$ cannot be convex for μ small enough; conversely, when $\mu = 1$, the injectivity domain of any point (including equatorial ones) are circles of radius π (the cut locus of any point on the round sphere is the antipodal point, at distance π), therefore convex. There is so some threshold between $\mu = 1$ and $\mu = 0$ regarding convexity. Besides, for any fixed $\mu \in [0, 1]$, the injectivity domains of poles are circles (as on the round sphere), and the same must hold for initial conditions $\varphi_0 \in (0, \pi/2)$ close enough to 0 (by symmetry, one can restrict to $\varphi < \pi/2$). See Fig. 4. The following is proved in [4,5].

Theorem 1. *There is a nondecreasing function* $\varphi_0 \mapsto \mu(\varphi_0)$ *from* $[0, \pi/2]$ *to* $\mathbf{R}$ *such that the injectivity domain* $I(\varphi_0)$ *on an oblate ellipsoid of semi-minor axis* $\mu \le 1$ *is convex if and only if* $\mu \ge \mu(\varphi_0)$*. One has* $\mu(0) = 0$ *(pole) and* $\mu(\pi/2) = 1/\sqrt{3}$ *(equator).*

The proof makes an essential use of the two following facts: (i) the degeneracy at $p_\theta = \pm 1$ of the elliptic curve to obtain $\mu(\pi/2) = 1/\sqrt{3}$; (ii) the fact that the cut time is given by the period of φ (this is not true anymore in the prolate case, see §3), which allows to derive analytic expressions of τ' and τ'' using derivatives of the periods of Weierstraß functions with respect to their invariants. As the threshold $\mu(\varphi_0)$ is monotonic, injectivity domains of any point on an oblate ellipsoid with $1/\sqrt{3} \le \mu \le 1$ are convex. The determination of $\mu(\varphi_0)$ for $\varphi_0 \in (0, \pi/2)$ is an open problem. (Numerical estimates are available, though.)

When $\mu = 0$, since crossing the boundary is changing hemisphere, one can also interpretate the geodesic continuing on the other side as a reflection on $\partial\mathbf{D}$ (with the usual rule on the angles). As a result, the conjugate locus is obtained as a catacaustic of the circle (see Fig. 3).

Proposition 4. *For* $\mu = 0$ *and* $\varphi_0 = \pi/2$*, the conjugate locus is a cardioid deprived of the initial point. The nonfocal domain is the union of two disks of radii* $4/3$ *both tangent to the* x*-axis at the origin, and is not convex (Fig. 2).*

Proof. The catacaustic of the unit circle with a source on the boundary is known to be the cardioid $z(\beta) := (2/3)(1 + \cos\beta)\exp(i\beta) - 1/3$ (see [2]). To prove that the associated nonfocal domain is the union of two circles, consider the ray generated by some $\alpha \in (0, \pi)$ in the parameterization (1) (considering $\alpha \in (-\pi, 0)$, that is p_{φ_0} negative, gives the other disk); it is enough to check that $w(\alpha) := 1 + 2\cos\alpha\exp(i(\pi - \alpha)) + (2/3)\cos\alpha\exp(-3i\alpha)$ (see construction on Fig. 3) belongs to the cardioid, which is clear. □

Remark 2. When $\mu = 0$, the metric is conformal to an almost-Riemannian metric with a singularity at the equator since

$$X\mathrm{d}\theta^2 + (1 - X)\mathrm{d}\varphi^2 = (1 - X)(XR(X)\mathrm{d}\theta^2 + \mathrm{d}\varphi^2)$$

with $R(X) = 1/(1 - X)$ having a pole of order one at $X = 1$ ($\varphi = \pi/2$). Such metrics are particular cases of sub-Riemannian metrics and are considered in [1,3]. Here, the conformal coefficient itself is singular, but the analysis is obvious because of the flatness of the metric. Note that the cut locus of an equatorial point is the equator minus a point, and that the contact of the conjugate locus with the initial point is of order one (compare Theorem 1 and 2 in [3]).

As for the injectivity domain, there exists some threshold phenomenon for the loss of convexity of the nonfocal domain of $\varphi_0 = \pi/2$ when μ goes to zero (see Fig. 2). Conversely, for a fixed $\mu \le 1$, convexity of nonfocal domains is retrieved when φ_0 tends to zero (see Fig. 4). Although numerical investigation suggests that some result similar to Theorem 1 may hold for nonfocal domains, the problem is open.

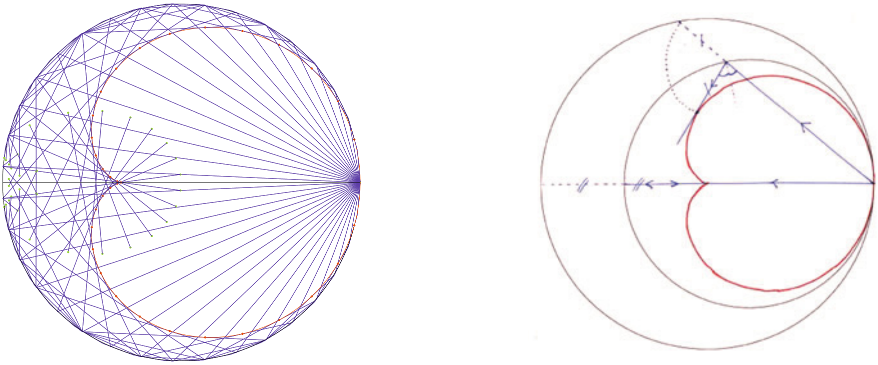


Fig. 3 Conjugate locus and nonfocal domain for $\mu = 0$ and $\varphi_0 = \pi/2$. On the disk, geodesics are straight lines starting from a point on the boundary and crossing $\partial\mathbf{D}$ when changing hemisphere can be seen as reflecting on $\partial\mathbf{D}$. The envelope of reflected rays forms the conjugate locus obtained as a catacaustic of the circle with source point on its boundary (leftmost graph). The geometric construction of the nonfocal domain from the cardioid is illustrated on the righmost picture

3 Prolate case

When $\mu > 1$, some loss of symmetry occurs, except when $X_0 = 1$.

Lemma 2. *The cut time along a geodesic (not a meridian) is obtained solving $\theta = \pi$. As such, $\tau = \tau(p_\theta, \mu)$ is independent of X_0 but depends on the sign of p_{φ_0}. The injectivity domain has just one axial symmetry wrt. y-axis, and convexity can be checked on a half of the domain.*

Proof. The situation in the prolate case is reversed compared to the oblate one: The symmetry $p_\theta \mapsto -p_\theta$ now generates intersections between geodesics emanating from the same point at length shorter than those generating by $p_{\varphi_0} \mapsto -p_{\varphi_0}$. Along every geodesic not a meridian, the cut is thus obtained at $\theta = \pi$ (while the meridian case is obtained as an envelope, letting p_θ tend to 0, providing a point both in the cut and conjugate loci). Clearly, $\pm p_\theta$ provide the same cut time, so the symmetry wrt. the y-axis of the injectivity domain is preserved. On the contrary, for $X_0 \neq 1$, geodesics with same p_θ but opposite p_{φ_0} do not cross $\theta = \pi$ at the same time, so that τ actually depends on the sign of p_{φ_0}; it has to be thought of as a function ramified above p_{φ_0} when parameterizing by p_θ alone. To prove that $\tau = \tau(p_\theta, X_0, \mu)$ actually does not depend on the initial condition, define $\Delta\theta$ the quasi-period of θ, that is the increment from $\theta_0 = 0$ on a period of φ. (According to Proposition 1, the period of X, and so of φ, is given in the complex parameterization by the imaginary period of the lattice). As in the oblate case, the period of φ only depends on p_θ, and so does $\Delta\theta$. Given $p_\theta > 0$, as $\mathrm{d}t/\mathrm{d}\theta = 1/\dot\theta = X/p_\theta > 0$, one can reparametrize using θ instead of t; since $\dot\theta$ and φ have the same period, X remains periodic as a

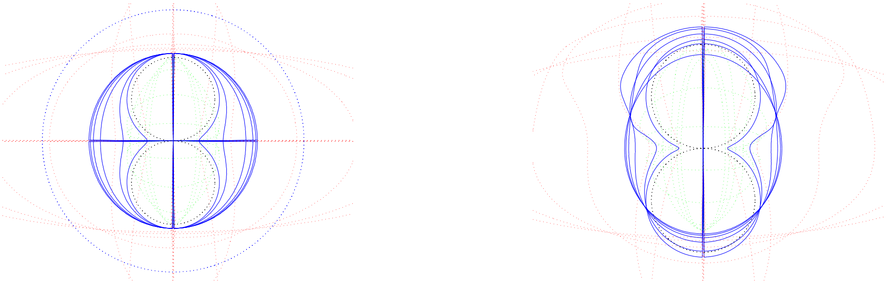


Fig. 4 Injectivity and nonfocal domains (left and right, respectively) for $\mu < 1/\sqrt{3}$ when $\varphi_0 \to \mathbf{0}$. For $\varphi_0 = \mathbf{0}$, both domains are disks, though for $\varphi_0 = \pi/2$ none are convex. Observe the loss of the axial symmetry wrt. the x-axis for the nonfocal domain

function of θ (with period $\Delta\theta(p_\theta)$), and

$$\tau = \int_0^\pi \frac{X(\theta)}{p_\theta}\,\mathrm{d}\theta.$$

Let $t_1 > 0$ be the first intersection of the geodesic with $\varphi = \pi/2$, assuming for the sake of simplicity $\varphi_0 < \pi/2$ and $p_{\varphi_0} > 0$ (the same kind of argument works for $p_{\varphi_0} < 0$). The geodesic of initial condition $(\theta(t_1), \pi/2)$ with same p_θ (and positive p_{φ_0}) has cut time

$$\tilde{\tau} = \int_{\theta(t_1)}^{\theta(t_1)+\pi} \frac{X(\theta)}{p_\theta}\,\mathrm{d}\theta = \int_0^\pi \frac{X(\theta)}{p_\theta}\,\mathrm{d}\theta$$

by periodicity of $X(\theta)$, so $\tau = \tilde{\tau}$, cut time associated with initial condition $\pi/2$, whatever φ_0. $\square$

Up to translation, X is given by some Weierstraß function, $\wp$, whose invariants depend only on p_θ and λ (that is on p_θ and μ – see Proposition 1). In the parameterization by $z \in \mathbf{C}/\Lambda$, one checks that the resulting quadrature on θ involves integrating rational fractions in $\wp$ such as

$$\int \frac{\wp'(a)\,\mathrm{d}z}{\wp(z) - \wp(a)} = 2\zeta(a)z + \ln\frac{\sigma(z-a)}{\sigma(z+a)}$$

where $\zeta' = -\wp$ and $\sigma'/\sigma = \zeta$. Studying the roots of an equation with such transcendence is a complicated task. We provide a preliminary analysis trying to take advantage of the degeneracy for $\mu = \infty$ when $p_\theta = 0$, and using numerical estimates.

Proposition 5. *The metric of the prolate ellipsoid converges pointwisely outside poles to the metric of the flat cylinder of revolution when $\mu \to \infty$. All injectivity and nonfocal domains of the cylinder are convex.*

Proof. Recalling that $z = \mu \cos\varphi$, the metric on the ellipsoid writes

$$\left(1 - \frac{z^2}{\mu^2}\right) \mathrm{d}\theta^2 + \left(1 + \frac{z^2}{\mu^2(\mu^2 - z^2)}\right) \mathrm{d}z^2$$

and convergence is clear. The geodesics on the cylinder of revolution are either vertical lines ($p_\theta = 0$) without cut points, or helices ($\mathrm{d}z/\mathrm{d}\theta = p_z/p_\theta = \mathrm{cst}$); in the second case, the cut time is $\pi/|p_\theta|$ (cut point on the antipodal vertical line). Injectivity domains are therefore all equal to a vertical strip $[-\pi, \pi] \times \mathbf{R}$, and convex. The metric is flat and there are no conjugate points, so nonfocal domains are the whole fiber ($\simeq \mathbf{R}^2$) at any point, also convex. □

In contrast with the oblate case, another complication is so that there is no obvious obstruction to convexity arising from the asymptotic behavior when $\mu \to \infty$. With implicit function use on $\theta = \pi$ in mind, we recall the computation of the sensitivities wrt. initial condition of first (Jacobi fields) and second order for a Hamiltonian system.

Remark 3. The fact that the metric converges towards a flat metric (previous Proposition) does not even entail that the limit, after some compactification, of the nonfocal domains must be convex (see Fig. 5).

Let $\dot{z} = \overrightarrow{H}(z)$ be a smooth Hamiltonian system, with $z = (x, p) \in \mathbf{R}^{2n}$ and $\overrightarrow{H} = (\partial_p H, -\partial_x H)$. The solution $z(\cdot, z_0)$ with initial condition $z(0) = z_0$ depends smoothly on z_0, and for any $\delta z_0 \in \mathbf{R}^{2n}$ one has

$$\frac{\partial z}{\partial z_0}(t, z_0)\delta z_0 = \delta z(t), \quad \frac{\partial^2 z}{\partial z_0^2}(t, z_0)(\delta z_0, \delta z_0) = \delta_2 z(t),$$

where δz and $\delta_2 z$ are solutions of, respectively (by $H[t]$ we mean $H(z(t, z_0))$, *etc.*),

$$\dot{\delta z} = \overrightarrow{H}'[t]\delta z, \quad \delta z(0) = \mathrm{id},$$

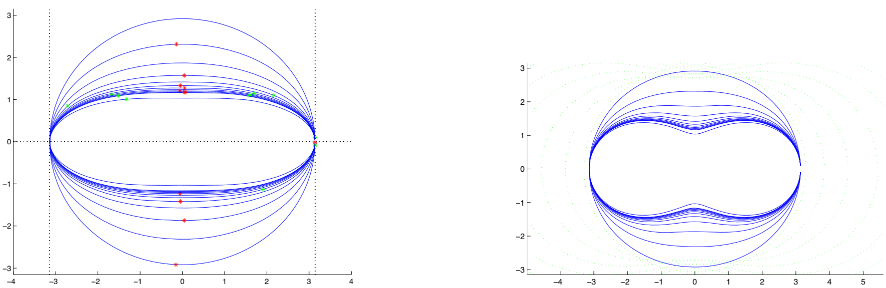
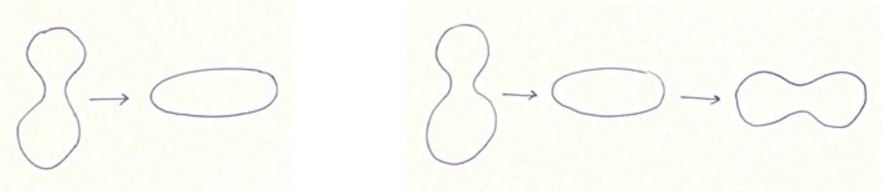


Fig. 5 Injectivity and nonfocal domains (left and right, respectively) of $\varphi_0 = \pi/2$ in the prolate case when $\mu \to \infty$. While convexity seems to hold at $\mu = \infty$ (and before) for the injectivity domain, nonfocal domains are clearly not convex for μ large enough, suggesting a threshold phenomenon as in the oblate case when $\mu \to \mathbf{0}$



$\mu < \mu(\varphi_0)$ $\quad$ $\mu(\varphi_0) \le \mu$ $\qquad$ $\mu < \tilde{\mu}(\varphi_0)$ $\quad$ $\tilde{\mu}(\varphi_0) \le \mu \le \hat{\mu}(\varphi_0)$ $\quad$ $\hat{\mu}(\varphi_0) < \mu$

Fig. 6 Conjecture on the bifurcation of convexity of injectivity and nonfocal domains on the ellipsoid of revolution for a given point (φ_0) not a pole when μ goes from $\mathbf{0}$ to ∞. Leftmost graph: For the injectivity domain, there might be only be one threshold $\mu(\varphi_0) < \mathbf{1}$. Rightmost graph: For the nonfocal domain, there might be two thresholds $\tilde{\mu}(\varphi_0) < \mathbf{1}$ and $\hat{\mu}(\varphi_0) > \mathbf{1}$, as convexity might be retrieved for μ close to $\mathbf{1}$, and lost again for μ large enough

$$\dot{\delta_2 z} = \vec{H}'[t]\delta_2 z + \vec{H}''[t](\delta z(t), \delta z(t)), \quad \delta_2 z(0) = 0.$$

The numerical computation of these sensitivities, up to order two, is performed by the `cotcot` software[2] combining automatic differentiation and numerical integration of ordinary differential equations. In our case, $z = (\theta, \varphi, p_\theta, p_\varphi)$ and, for $p_\theta \neq 0$, τ is implicitely defined by $\theta(\tau, p_\theta) = \pi$. As previously mentioned, there is a dependence of the geodesic not only on p_θ but also on the sign of p_{φ_0}. The initial condition on $\{H = 1/2\}$ writes

$$z_0(p_\theta) := \left(p_\theta, \pm\sqrt{1 - X_0/\lambda}\,\sqrt{1 - p_\theta^2/X_0}\right).$$

Proposition 6. *For $0 < p_\theta^2 < X_0$, the derivatives of first and second order are*

$$\tau' = -\frac{1}{\dot{\theta}}\delta\theta, \quad \tau'' = -\frac{1}{\dot{\theta}}(\ddot{\theta}\tau'^2 + 2\delta\dot{\theta}\tau' + \delta_2\theta + \widetilde{\delta\theta}),$$

where $\delta\theta$ (resp. $\delta_2\theta$) is the first (resp. second) variation associated with $\delta z_0 = z_0'(p_\theta)$, $\widetilde{\delta\theta}$ the first variation associated with $\widetilde{\delta z_0} = z_0''(p_\theta)$, and where all functions are evaluated at $\tau(p_\theta)$.

Proof. Apply implicit function theorem to $\theta(\tau, p_\theta) = \pi$ noting that $\dot{\theta} = p_\theta/X \neq 0$ whenever $p_\theta \neq 0$. □

Whereas the worst case for curvature on an oblate ellipsoid, whatever the point, is given by the equator ($p_\theta^2 = X_0$), numerical simulations below indicate that the worst case in the prolate situation is given by meridians, $p_\theta = 0$, at the apparent singularity of the expressions before. Worst cases for curvature of injectivity domains (and, seemingly, of nonfocal domains – see Fig. 5) actually occur along geodesics where cut points are conjugate ones (equator in the oblate case, meridian in the prolate one).

To achieve numerical convergence of the domains, and of the curvature, we use a second dilation: $(x, y) \mapsto (x, y/\mu)$. The curvature is thus renormalized according

[2] `apo.enseeiht.fr/cotcot`

to

$$\widetilde{K} = (1/\mu)X_0^{3/2}\frac{\tau(\tau + p_\theta\tau') + (X_0 - p_\theta^2)(2\tau'^2 - \tau\tau'')}{[(X_0 - p_\theta^2)(\tau + p_\theta\tau')^2 + (1/\mu)^2(p_\theta\tau - (X_0 - p_\theta^2)\tau')^2]^{3/2}}.$$

This provides a heuristical compactification of domains and curvature, but has the effect that the parameterization by $p_\theta = \cos\alpha\sqrt{X_0}$ becomes singular when $\mu \to \infty$ as

$$\alpha = \arg\left(\frac{p_\theta}{\sqrt{X_0}} + i\frac{p_{\varphi 0}}{\sqrt{1 + (\mu^2 - 1)X_0}}\right) \to 0$$

whenever $p_\theta \neq 0$. One parameterizes instead $\widetilde{K}$ using $\beta := \arg(\cos\alpha + (i/\mu)\sin\alpha)$. On the basis of numerical estimates computed as in Proposition 6, the following observations can be made: (i) For $\varphi_0 = \pi/2$, numerical convergence of the (renormalized) injectivity domain is obtained (see Fig. 5); the limit domain seems to be convex, which suggests that convexity holds for equatorial points and μ large enough. A stronger conjecture would be convexity for all $\mu > 1$, or even for all $\mu > 1$ whatever φ_0 (see also Fig. 8 in this respect). (ii) For $\varphi_0 = \pi/2$, an estimation of the curvature $\widetilde{K}$ of the (renormalized) injectivity domain is obtained (see Fig. 7), not contradicting (i). (iii) For $\varphi_0 = \pi/2$, numerical convergence of the (renormalized) nonfocal domain is also obtained (see Fig. 5), which suggests that convexity does not hold for large enough μ; one can conjecturate a threshold phenomenon as in the oblate situation. (iv) The dependence of the convexity on the initial condition for $\mu > 1$ seems to be more complicate than in the oblate case, both for injectivity and nonfocal domains, as no monotonic behaviour seems to hold (see Fig. 8). For a fixed φ_0, Fig. 6 summarizes the previous conjectures on the bifurcation of the domains in terms of convexity.

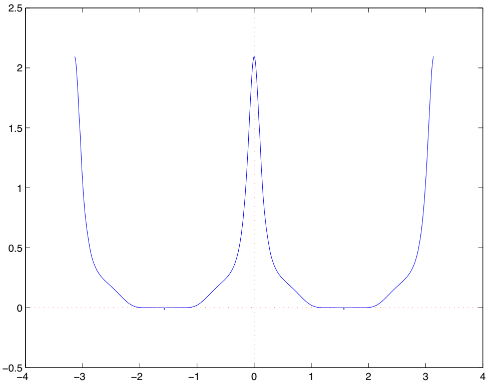


Fig. 7 Renormalized curvature $\widetilde{K}$ of the injectivity domain for $\mu = \infty$ and $\varphi_0 = \pi/2$. The parameter in abscissa is $\beta = \arg(\cos\alpha + (i/\mu)\sin\alpha)$ with $p_\theta = \cos\alpha\sqrt{X_0}$ so meridians are retrieved for $\beta = \pm\pi/2$. They actually correspond to the minimum estimated value of the curvature, in accordance with Fig. 5

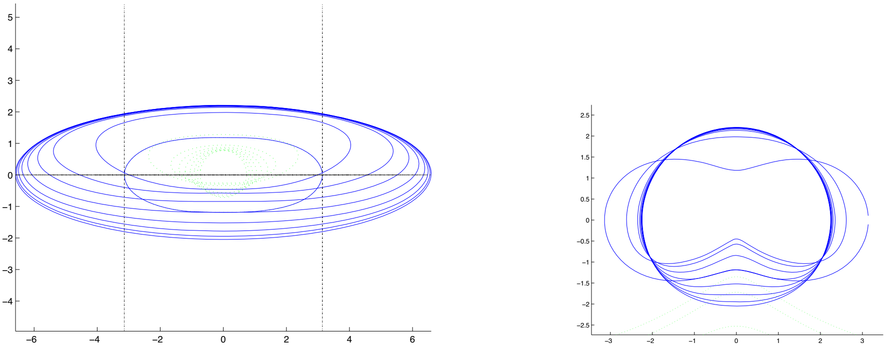


Fig. 8 Injectivity and nonfocal domains (left and right, respectively) for $\mu > 1$ when $\varphi_0 \to 0$. For $\varphi_0 = 0$, both domains are disks; for $\varphi_0 = \pi/2$, the injectivity domain remains convex but not the nonfocal domain. For $\varphi_0 \in (0, \pi/2)$, domains only have one axial symmetry. Monotonic dependence of the curvature on φ_0 does not seem to hold, either for the injectivity domain, or for the nonfocal one

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