NON-INTEGRABILITY OF THE MINIMUM TIME KEPLER PROBLEM

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Abstract. We prove that the minimum time controlled Kepler problem is not meromorphically Liouville integrable on the Riemann surface of its Hamiltonian.

1. Introduction

The Kepler problem

$$\ddot{q} + \frac{q}{\|q\|^3} = 0, \quad q \in \mathbb{R}^2 \setminus \{0\}. \tag{1}$$

is a classical reduction of the two-body problem [2]. Here, we think of q as the position of a spacecraft, and of the attraction as the action of the Earth. We are interested in controlling the transfer of the spacecraft from one Keplerian orbit towards another, in the plane. Denoting $v = \dot{q}$ the velocity, and the adjoint variables of q and v by p_q and p_v, the minimum time dynamics is a Hamiltonian system with Hamiltonian

$$H(q, v, p_q, p_v) = p_q.v - \frac{p_v.q}{\|q\|^3} + \|p_v\|, \tag{2}$$

as is explained in section 2.1. Prior studies of this problem can be found in [4, 6]. The controlled Kepler problem can be embedded in the two parameter family obtained when considering the control of the circular restricted three-body problem:

$$\ddot{q} + \nabla_q \Omega_\mu(t, q) = \varepsilon u, \tag{3}$$

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where

$$\begin{aligned}\Omega_\mu(t,q) &= -\frac{1-\mu}{\sqrt{(q_1+\mu\cos t)^2+(q_2+\mu\sin t)^2}}\\ &\quad -\frac{\mu}{\sqrt{(q_1-(1-\mu)\cos t)^2+(q_2-(1-\mu)\sin t)^2}}\end{aligned}$$

is the potential parameterized by the ratio of masses, $\mu \in [0,1/2]$, and where $u \in \mathbb{R}^2$ is the control, whose amplitude is modulated by the second parameter, $\varepsilon \geq 0$. Alternatively to time minimization, minimization of the $\mathbb{L}^2$ norm of the control can be considered,

$$\int_0^{t_f} u^2(t)\,\mathrm{d}t \to \min\,.$$

This is the so-called energy cost. In the uncontrolled case ($\varepsilon = 0$), it is well known that the Kepler case ($\mu = 0$) is integrable (and geodesic) while there are obstructions to integrability for positive μ. In the controlled case ($\varepsilon > 0$), the Kepler problem for the energy cost has been shown to be integrable (and geodesic) when suitably averaged (see [5] for a survey). The aim of this paper is to study the integrability properties of the Kepler problem for time minimization.

The pioneering work of Ziglin in the 80's [18], followed by the modern formulation of differential Galois theory in the late 90's by Moralès, Ramis and Simó [13, 12], have led to a very diverse literature on the integrability of Hamiltonian systems. According to Pontrjagin's Maximum principle, one can turn general optimization problems with dynamical constraints into Hamiltonian systems, which are generally not everywhere differentiable. Optimal control theory thus provides an abundant class of dynamical systems, for which integrability naturally is a central question. Yet, differential Galois theory has not so often been applied in this context, in part because of the difficulty brought by the singularities. Notwithstanding theses singularities (vanishing of the adjoint variable p_v, here), we show how to apply these ideas to the system (2).

2. Setting

2.1. **The minimum time controlled Kepler problem.** We first recall some classical facts on optimal control. We refer for example to the book of Agrachev and Sachkov [1] for more details. Let M be an n-dimensional smooth manifold and U an arbitrary subset of $\mathbb{R}^m$ (typically a submanifold with boundary). A controlled dynamical system is a smooth family of vector fields

$$f : M \times U \to TM$$

parameterized by the control values. Admissible controls are measurable functions valued in the subset U. A preliminary question is the following: Is some final state x_f accessible from some initial state x_0, *i.e.* does the system

$$\dot{x}(t) = f(x(t), u(t)), \quad u(t) \in U,$$

$$x(0) = x_0, \quad x(t_f) = x_f,$$

have a solution for some control u in $\mathcal{U}$? The system is said to be controllable if the answer is positive for all possible initial and final states $x_0, x_f \in M$. The controlled Kepler problem, associated with (1), is

$$\ddot{q} + \frac{q}{\|q\|^3} = u, \quad q \in \mathbb{R}^2 \setminus \{0\}, \quad u_1^2 + u_2^2 \leq 1,$$
$$(q(0), \dot{q}(0)) = (q_0, v_0), \quad (q(t_f), \dot{q}(t_f)) = (q_f, v_f),$$

where q is the position vector of a spacecraft and where the control u is the thrust of the engine. The thrust is obviously bounded; here we assume that it takes values in the euclidean unit ball. (Note that, with respect to (3), we have chosen $\varepsilon = 1$; as will be clear from Section 3, this does not restrict the generality of the analysis.)

Proposition 1 ([6]). *The Kepler problem is controllable.*

This is a consequence of two facts: The Lie algebra generated by the drift and the vector field supporting the control generate the whole tangent space in each point (which entails some local controllability), and the uncontrolled flow (or *drift*) of the Kepler problem is recurrent. Under some additional compactness assumptions, one is then able to retrieve existence of optimal controls.

We now deal with such optimal controls. Here we will restrict ourselves to integral cost functions, that is to problems of the form

$$\begin{cases} \dot{x}(t) = f(x(t), u(t)), \\ x(0) = x_0, \quad x(t_f) = x_f, \\ \int_0^{t_f} L(x(t), u(t))\, \mathrm{dt} \to \min \end{cases} \tag{4}$$

where the final time t_f can be fixed or not, and $L : M \times U \to \mathbb{R}$ is a smooth function. In the early 60's, Pontrjagin and his coauthors realized that necessary conditions for optimality could be stated in Hamiltonian terms. By T^*M we denote the cotangent bundle of the manifold M.

Definition 1. The associated pseudo-Hamiltonian is

$$H : T^*M \times \mathbb{R} \times U \to \mathbb{R}, \quad (x, p, p^0, u) \mapsto \langle p, f(x, u)\rangle + p^0 L(x, u).$$

The following fundamental result is Pontrjagin Maximum Principle [15] (see [1] for a modern presentation).

Theorem 1. *If* (x, u) *solves* (4)*, there exists an absolutely continuous (actually Lipschitzian) function* $p(t) \in T^*_{x(t)}M$*,* $t \in [0, t_f]$*, a constant* $p^0 \leq 0$*, such that* $(p(t), p^0) \neq 0$ *and such that, almost everywhere,*

(i) (x, p) *is a solution of the Hamiltonian system associated with* $H(\cdot, \cdot, u(t))$*:*

$$\dot{x} = \frac{\partial H}{\partial p}(x, p, u), \quad \dot{p} = -\frac{\partial H}{\partial x}(x, p, u),$$

(ii) $H(x(t), p(t), u(t)) = \max_{v \in U} H(x(t), p(t), v)$.

Such curves (x, p) are called extremals. As a consequence of the maximization condition, the pseudo-Hamiltonian evaluated along an extremal is constant. Moreover, if the final time is free then this constant is zero.

This powerful result has some downsides. The Hamiltonian is defined on the cotangent bundle of the original phase space, and thus the dimension is doubled. Besides, the maximization condition, which "eliminates the control" and allows to obtain a truly Hamiltonian system in (x, p) only, generates singularities (that is non-differentiability points of the maximized Hamiltonian which is, whenever defined, only Lipschitzian in general). The above theorem applies to time minimization with $L \equiv 1$ (and free final time). In this case, the non-positive constant p^0 is only related to the level of the Hamiltonian, and we will not mention it in the sequel as we will not discuss the implications of having normal ($p^0 \neq 0$) or abnormal ($p^0 = 0$) extremals.

2.2. **Main result.** The minimum time Kepler problem can be stated according to

$$\begin{cases} \ddot{q} + \frac{q}{\|q\|^3} = u, \quad \|u\| \leq 1, \\ (q(0), \dot{q}(0)) = (q_0, v_0), \quad (q(t_f), \dot{q}(t_f)) = (q_f, v_f), \\ t_f \to \min, \end{cases} \tag{5}$$

where as before $q \in \mathbb{R}^2$ is the position vector and $u \in \mathbb{R}^2$ the control. It will be convenient to use the same notation as in the general problem (4), and let

$$q = (x_1, x_2), \quad \dot{q} = (x_3, x_4),$$

be the coordinates on the initial phase space $M = (\mathbb{R}^2 \setminus \{0\}) \times \mathbb{R}^2$. According to Definition 1, the pseudo-Hamiltonian is then

$$H(x, p, u) = p_1 x_3 + p_2 x_4 - \frac{p_3 x_1 + p_4 x_2}{(x_1^2 + x_2^2)^{3/2}} + p_3 u_1 + p_4 u_2.$$

According to Theorem 1, minimizing trajectories must be projections on M of integral curves of the Hamiltonian that has to be maximized over the unit disk. Clearly, the maximized Hamiltonian is equal to

$$H(x, p) = p_1 x_3 + p_2 x_4 - \frac{p_3 x_1 + p_4 x_2}{(x_1^2 + x_2^2)^{3/2}} + \sqrt{p_3^2 + p_4^2}$$

on T^*M, while the control is given by

$$u = \frac{1}{\sqrt{p_3^2 + p_4^2}} (p_3, p_4)$$

whenever p_3 and p_4 do not vanish simultaneously. (This will indeed be the case for the type of extremals we will consider in the rest of the paper.) Now, let

$$\mathcal{M} = \{(x, p, r) \in \mathbb{C}^8 \times \mathbb{C}_*^2,\ r_1^2 = x_1^2 + x_2^2,\ \ r_2^2 = p_3^2 + p_4^2\}$$

be the Riemann surface of H. It is a complex symplectic manifold (with local Darboux coordinates (x,p) outside the singular hypersurface $r_1 r_2 = 0$), over which H extends meromorphically, and even rationally, since

$$H(x,p,r) = p_1 x_3 + p_2 x_4 - \frac{p_3 x_1 + p_4 x_2}{r_1^3} + r_2. \tag{6}$$

The Hamiltonian H has four degrees of freedom, hence (see [2]) the meromorphic Liouville integrability of H over $\mathcal{M}$ would mean that there would exist three independent first integrals, in addition to H itself, almost everywhere in $\mathcal{M}$. This does not happen.

Theorem 2. *The minimum time Kepler problem is not meromorphically Liouville integrable on $\mathcal{M}$.*

It is well known that the classical Kepler problem is integrable, and even super integrable (since there are more first integrals than degrees of freedom, which relates to Kepler's first law and the dynamical degeneracy of the Newtonian potential see [8], for example). But the three-body problem is not, as Poincaré, Julliard-Tosel, Tsygvintsev or Combot have proved in various settings [7, 9, 14, 17]. Similarly, the above theorem asserts that lifting the Kepler problem to the cotangent bundle and introducing the singular control term r_2 break integrability.

This result prevents the existence of enough real analytic (and even meromorphic) first integrals to ensure integrability over $\mathcal{M}$. Or course, it does not prevent the existence of an additional first integral which would have a natural frontier outside asymptotic to the real domain and which would thus not extend to the complex plane. Future work may be dedicated to extend this result to real first integrals: The Hamiltonian is invariant by two rotations, and homogeneous with respect to several coordinates. If it is real Liouville integrable, one can find real first integrals have the same symmetries. This may allow us to extend them to the complex domain, creating a contradiction with theorem 2.

3. Proof of theorem 2

The rest of the article is devoted to proving the theorem. Our proof consists in studying the variational equation along some integral curve of (6). In order to carry out this computation, we choose a collision orbit, with the drawback that it requires some regularization. We also note that there exist effective tools to perform this kind of computations (see, *e.g.*, [11]).

3.1. **Some facts of Galois differential theory.** The algebraic obstruction to Liouville integrability will come from the theorem below of Moralès and Ramis, which we now recall. Consider a linear differential equation $(L) : Y' = AY$, $A \in M_n(k)$, k being an algebraically closed differential field.

Definition 2 (Picard-Vessiot field)**.** The Picard-Vessiot field of (L)—denoted K—over the field k is the smallest extension of k containing the component of the solutions of (L).

Note that if $u_1, \ldots, u_n$ is a fundamental system of solutions of (L), then $K = k(u_{11}, \ldots, u_{nn})$. We can now define the Galois group of a linear differential equation.

Definition 3 (Galois group)**.** The Galois group of the Picard-Vessiot extension K over k, denoted $Gal(K/k)$, is the group of differential automorphisms of K preserving the set of solutions of (L) and which commutes with the derivation.

If there is no ambiguity, we use the notation : $Gal(A)$. As the automorphisms are differential, i.e. they should commute with the derivation, we see that the action of such automorphism stabilizing k on a solution u of L should also give a solution of L. Thus an automorphism of the Galois group can be identified as a linear transformation on a basis of solutions of L, i.e. an invertible matrix. Thus the Galois group can be identified with a subgroup of $Gl_n(\mathbb{C})$. It appears that such groups need to be algebraic Lie groups. We are interested in non integrability for Hamiltonian systems, giving, in general, non linear differential equations. The link with Hamiltonian is given by the now famous Moralés-Ramis theorem below. We recall that a group G is said to be virtually Abelian if its connected component containing the identity is an Abelian subgroup of G.

Theorem 3 (Moralès-Ramis [12])**.** *Let us consider a Hamiltonian H analytic on a complex analytic symplectic manifold and a non constant solution Γ. If H is integrable in the Liouville sense with meromorphic first integrals, then the first order variational equation along Γ has a virtually Abelian Galois group over the base field of meromorphic functions on Γ.*

The main idea behind this theorem is that if H is Liouville integrable, then so are the linearized equations near a non constant solution Γ. More precisely, thanks to Ziglin's Lemma, the first integrals of H can be transformed such that their first non trivial term in their series expansion near Γ are functionally independant. Now these first terms are commuting independant first integrals of the variational equation near Γ. In their paper, Moralès-Ramis [12] precisely proved that the Galois group of symplectic linear differential systems having such first integrals have a Galois group whose identity component is Abelian. This result can be expected knowing that the Galois group leaves invariant every first integral, and thus the more first integrals, the smaller Galois gropup.

We will need later on the definition of the monodromy group of a linear differential equation. It is the matrix subgroup generated by the action on the set of solutions, of the fundamental group of the complex domain without the singularities. Its most useful property for our proof is that it is included in the Galois group (see the Schlesinger density theorem below), and thus that if it is non Abelian result then the system is non integrable.

Theorem 4 (Schlesinger density theorem [16])**.** *Let* $(E):\ Y' = AY$ *be a Fuchsian differential linear equation with coefficients in* $\mathbb{C}(x)$ *and let* Π *be its monodromy group. Then* Π *is dense for the Zariski topology in the Galois group of the Picard-Vessiot extension of* (E) *over the base field of rational functions:* $\bar{\Pi} = Gal_{\mathbb{C}(x)}(A)$.

3.2. **A collision orbit.** In order to find an explicit solution, let us define the 4-dimensional symplectic submanifold

$$S = \{(x,p,r), \quad x_2 = x_4 = p_2 = p_4 = 0, r_1 = x_1, r_2 = -p_3\} \cap \mathcal{M}.$$

As S is the phase space of the controlled Kepler problem on the line (collision orbit) parameterized by q_1, it is invariant. On the interior of S, (x_1, x_3, p_1, p_3) is a set of (Darboux) coordinates and, in restriction to S, the Hamiltonian reduces to

$$H(x,p) = p_1 x_3 - \frac{p_3}{x_1^2} - p_3,$$

so the Hamiltonian vector field on S is

$$\begin{cases} \dot{x}_1 = x_3 \\ \dot{x}_3 = -1 - \frac{1}{x_1^2} \\ \dot{p}_1 = -\frac{2p_3}{x_1^3} \\ \dot{p}_3 = -p_1. \end{cases}$$

In particular,

$$\begin{cases} \ddot{x}_1 = -1 - \frac{1}{{x_1}^2} \\ \ddot{p}_3 - \frac{2p_3}{x_1^3} = 0. \end{cases} \tag{7}$$

As is known since the work of Charlier and Saint Germain on the Kepler problem with a constant force (see [3]), the function

$$C = \frac{1}{2}x_3^2 + x_1 - \frac{1}{x_1}$$

is a first integral on S and $H_{|S}$ is integrable. Let us switch to the time $s = x_1(t)$, in which an explicit solution will be written, and denote by $' = \frac{d}{ds}$ derivatives with respect to the new time. It suffices to find an obstruction in this modified time, as explained at the end of the proof.

Using (7), we see that the variable p_3, as a function of the new time, satisfies the linear differential equation

$$2\left(C + \frac{1}{x_1} - x_1\right) p_3''(x_1) - \left(1 + \frac{1}{x_1^2}\right) p_3'(x_1) - \frac{2p_3(x_1)}{x_1^3} = 0,$$

which yields

$$p_3(x_1) = \frac{\sqrt{-Cx_1 + x_1^2 - 1}}{\sqrt{x_1}} \left(c_1 \int \frac{x_1^{3/2}}{(-Cx_1 + x_1^2 - 1)^{3/2}} dx_1 + c_2\right)$$

for some constants of integration c_1 and c_2. Here the symbol $\int f(x_1)dx_1$ denotes some primitive of f with respect to the variable x_1. It suffices to find one particular integral curve along which the variational equation has a non virtually Abelian Galois group. To this end, we consider the simple—but rich enough—case when $c_1 = 0$, $c_2 = 1$.

$$p_3(x_1) = \frac{\sqrt{-Cx_1 + x_1^2 - 1}}{\sqrt{x_1}}.$$

Using the expression of the first integral C and of the vector field, we deduce

$$x_3(x_1) = \sqrt{2}\frac{\sqrt{-Cx_1 + x_1^2 - 1}}{\sqrt{x_1}}, \quad p_1(x_1) = -\frac{1}{\sqrt{2}}\frac{x_1^2 + 1}{x_1^2}.$$

Choosing $C = 2i$ and some determination of the squares yields a particularly simple solution Γ drawn on $S \subset \mathcal{M}$,

$$\begin{cases} x_1 = x_1 \\ x_2 = 0 \\ x_3 = \sqrt{2}\frac{x_1 - i}{\sqrt{x_1}} \\ x_4 = 0 \end{cases} \qquad \begin{cases} p_1 = -\frac{x_1^2+1}{\sqrt{2}x_1^2} \\ p_2 = 0 \\ p_3 = \frac{x_1-i}{\sqrt{x_1}} \\ p_4 = 0. \end{cases} \tag{8}$$

3.3. **Normal variational equation.** In the initial time, the linearized equation along Γ is the Hamiltonian vector field associated with the Hamiltonian DH along Γ:

$$\dot{Z}(t) = A(t)Z(t), \quad A(t) = J\,D^2H(\Gamma(t)),$$

where J is the Poisson structure. In the coordinates $(x_1, ..., x_4, p_1, ..., p_4)$,

$$J = \begin{pmatrix} 0_4 & I_4 \\ -I_4 & 0_4 \end{pmatrix}.$$

But we will keep using time x_1, instead of the initial time t, writing

$$Z'(x_1(t)) = \frac{1}{x_3(t)}A(x_1(t))Z(x_1(t)).$$

Let us now order coordinates as $(x_1, x_3, p_1, p_3, x_2, x_4, p_2, p_4)$. Since S is an invariant submanifold, the 8×8 matrix A has an upper triangular bloc structure

$$A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}$$

with

$$A_3 = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{\sqrt{2}p_3} \\ -\frac{1}{\sqrt{2}p_3^2} & 0 & -\frac{1}{\sqrt{2}x_1^3 p_3} & 0 \\ 0 & \frac{1}{\sqrt{2}p_3} & 0 & 0 \\ -\frac{1}{\sqrt{2}x_1^3 p_3} & 0 & \frac{3}{\sqrt{2}x_1^4} & 0 \end{pmatrix}.$$

The Morales-Ramis Theorem gives necessary conditions for Liouville integrability in terms of the Galois group of this linear differential system over the base field of meromorphic functions *on* Γ. Looking at the expression (8) of Γ, we see that meromorphic functions on Γ are just meromorphic functions in $\sqrt{x_1} \in \mathbb{C}\backslash\{0, \pm\sqrt{i}\}$. The block A_3 corresponds to infinitesimal variations in the normal direction to S, which is the part where interesting phenomena might occur. As the Picard-Vessiot field is generated by all the components of the solutions, the Picard Vessiot field K generated by the normal variational equation

$$(L): \quad X' = A_3 X, \quad X = (X_1, X_2, X_3, X_4)$$

is a subfield of the Picard Vessiot field of the whole variational equation, and thus $\mathrm{Gal}(A) \supset \mathrm{Gal}(A_3)$. That $\mathrm{Gal}(A_3)$ is not virtually Abelian will thus imply that $\mathrm{Gal}(A)$ itself is not virtually Abelian. In order to reduce the system to a one dimensional linear equation, we use the cyclic vector method on A_3: From (L) we get $X_1' = L_1(X_1, X_2, X_3, X_4)$, where L_1 is a linear form on $\mathbb{R}^4$, thus by derivation,

$$\begin{aligned} X_1'' =& L_1(X_1', X_2, X_3, X_4) + L_1(X_1, X_2', X_3, X_4) \\ &+ L_1(X_1, X_2, X_3', X_4) + L_1(X_1, X_2, X_3, X_4') \\ =& L_2(X_1, X_2, X_3, X_4). \end{aligned}$$

Iterating, we obtain

$$\begin{cases} X_1 = X_1 \\ X_1' = L_1(X_1, X_2, X_3, X_4) \\ X_1'' = L_2(X_1, X_2, X_3, X_4) \\ X_1^{(3)} = L_3(X_1, X_2, X_3, X_4) \\ X_1^{(4)} = L_4(X_1, X_2, X_3, X_4), \end{cases}$$

where the L_i's are linear forms on $\mathbb{R}^4$. These are five linear forms on $\mathbb{R}^4$, so X_1 must satisfy some linear differential equation of order 4 that we compute to be

$$\begin{aligned} (9) \quad X_1^{(4)} + \frac{2(3i - 5x_1)}{x_1(i - x_1)} X_1^{(3)} + \frac{(-3x_1 + i)(-29x_1 + 23i)}{4(x_1 - i)^2 x_1^2} X_1'' \\ - \frac{(i - 3x_1)(7x_1 + i)}{4(x_1 - i)^2 x_1^3} X_1' + \frac{3x_1 + i}{4(x_1 - 1)^3 x_1^4} X_1 = 0. \end{aligned}$$

We find a solution of this equation of the form

$$X_1(x_1) = \frac{i - x_1}{\sqrt{x_1}} \left(c_1 + c_2 \int \sqrt{x_1} (1 + i x_1)^{-\frac{3}{2} - i\frac{\sqrt{3}}{2}} . {}_2F_1(\gamma(x_1)) dx_1 \right),$$

where ${}_2F_1$ is the Gauss hypergeometric function and

$$\gamma(x_1) = \left(\frac{5}{2} - \frac{\sqrt{3}}{2} i, \frac{1}{2} + \frac{\sqrt{3}}{2} i, 1 + \sqrt{3} i, 1 + i x_1 \right).$$

The Picard-Vessiot field K contains this solution, and as it is a differential field, it also contains

$$\sqrt{x_1}(1+ix_1)^{-\frac{3}{2}-i\frac{\sqrt{3}}{2}}\,{}_2F_1(\gamma(x_1)).$$

Noting $\tilde{K}$ the differential field generated by this above function, we have $\tilde{K} \subset K$. Now the Galois group of ${}_2F_1(\gamma(x_1))$ over $\mathbb{C}(x_1)$ is $SL_2(\mathbb{C})$ (see Kimura's table, [10]). By Galois correspondance, the Galois group of 9 over the rational functions in x_1 has $SL_2(\mathbb{C})$ as a subgroup. The hypergeometric equation (9) is Fuchsian (meaning that all its singular points are regular), so thanks to Theorem 4, we know its Galois group over the field of rational functions is the closure of its monodromy group. Besides, the Galois group over meromorphic functions contains the monodromy group, and of course, is included in the Galois group over rational functions. Eventually, the Galois group of 9 over meromorphic functions in x_1 also contains $SL_2(\mathbb{C})$. Thus, adding the algebraic extension $\sqrt{x_1}$, the Galois group can be reduced to at most one subgroup of index 2, and thus there can only be $SL_2(\mathbb{C})$ again. So the Galois group of K over the base field of meromorphic functions in $\sqrt{x_1} \in \mathbb{C}\setminus\{0, \pm\sqrt{i}\}$ contains $SL_2(\mathbb{C})$ and is not virtually Abelian. According to Morales-Ramis, this concludes the proof.

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