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Non-integrability of the minimum-time Kepler problem[☆]

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ABSTRACT

We prove, using Moralès–Ramis theorem, that the minimum-time controlled Kepler problem is not meromorphically integrable in the Liouville sens on the Riemann surface of its Hamiltonian.

1. Introduction

The Kepler problem

$$\ddot{q} + \frac{q}{\|q\|^3} = 0, \quad q \in \mathbb{R}^2 \setminus \{0\}. \tag{1}$$

is a classical reduction of the two-body problem [1]. Here, we think of q as the position of a spacecraft, and of the attraction as the action of the Earth. We are interested in controlling the transfer of the spacecraft from one Keplerian orbit towards another, in the plane. Denoting $v = \dot{q}$ the velocity, and the adjoint variables of q and v by p_q and p_v, the minimum time dynamics is a Hamiltonian system with

$$H(q, v, p_q, p_v) = p_q.v - \frac{p_v.q}{\|q\|^3} + \|p_v\|, \tag{2}$$

as is explained in Section 2.1. Prior studies of this problem can be found in [2,3]. The controlled Kepler problem can be embedded in the two parameter family obtained when considering the control of the circular restricted three-body problem:

$$\ddot{q} + \nabla_q \Omega_\mu(t, q) = \varepsilon u, \tag{3}$$

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where

$$\Omega_\mu(t,q) = -\frac{1-\mu}{\sqrt{(q_1+\mu\cos t)^2+(q_2+\mu\sin t)^2}} - \frac{\mu}{\sqrt{(q_1-(1-\mu)\cos t)^2+(q_2-(1-\mu)\sin t)^2}}$$

is the potential parameterized by the ratio of masses, $\mu \in [0, 1/2]$, and where $u \in \mathbb{R}^2$ is the control, whose amplitude is modulated by the second parameter, $\varepsilon \geq 0$. Alternatively to time minimization, minimization of the $\mathbb{L}^2$ norm of the control can be considered,

$$\int_0^{t_f} u^2(t)\,\mathrm{d}t \to \min .$$

This is the so-called energy cost. In the uncontrolled model ($\varepsilon = 0$), it is well known that the Kepler case ($\mu = 0$) is integrable and geodesic (there exists a Riemannian metric such that Keplerian curves are geodesics of this metric [4,5]) while there are obstructions to integrability for positive μ. In the controlled case ($\varepsilon > 0$), the Kepler problem for the energy cost has been shown to be integrable (and geodesic) when suitably averaged (see [6] for a survey). The aim of this paper is to study the integrability properties of the Kepler problem for time minimization.

The pioneering work of Ziglin in the 80s [7], followed by the modern formulation of differential Galois theory in the late 90s by Moralès, Ramis and Simó [8,9], have led to a very diverse literature on the integrability of Hamiltonian systems. According to Pontryagin s Maximum principle, one can turn general optimization problems with dynamical constraints into Hamiltonian systems, which are generally not everywhere differentiable. Optimal control theory thus provides an abundant class of dynamical systems for which integrability is a central question. Yet, differential Galois theory has not so often been applied in this context (see, *e.g.*, [10]), in part because of the difficulty brought by the singularities. Notwithstanding theses singularities (vanishing of the adjoint variable p_v, here), we show how to apply these ideas to the system (2).

2. Setting

2.1. The minimum time controlled Kepler problem

We first recall some classical facts on optimal control. We refer for example to the book of Agrachev and Sachkov [11] for more details. Let M be an n-dimensional smooth manifold and U an arbitrary subset of $\mathbb{R}^m$ (typically a submanifold with boundary). A controlled dynamical system is a smooth family of vector fields

$$f : M \times U \to TM$$

parameterized by the control values. Admissible controls are measurable functions valued in the subset U. A preliminary question is the following: Is some final state x_f accessible from some initial state x_0, *i.e.* does the system

$$\dot{x}(t) = f(x(t), u(t)), \quad u(t) \in U,$$

$$x(0) = x_0, \quad x(t_f) = x_f,$$

have a solution for some admissible control? The system is said to be controllable if the answer is positive for all possible initial and final states $x_0, x_f \in M$. The controlled Kepler problem, associated with (1), is

$$\ddot{q} + \frac{q}{\|q\|^3} = u, \quad q \in \mathbb{R}^2 \setminus \{0\}, \quad u_1^2 + u_2^2 \leq 1,$$

$$(q(0), \dot{q}(0)) = (q_0, v_0), \quad (q(t_f), \dot{q}(t_f)) = (q_f, v_f),$$

where q is the position vector of a spacecraft and where the control u is the thrust of the engine. The thrust is obviously bounded; here we assume that it is valued in the Euclidean unit ball. (Note that, with respect to (3), we have chosen $\varepsilon = 1$; as will be clear from Section 3, this does not restrict the generality of the analysis.)

Proposition 1 (*[3]*). *The Kepler problem is controllable.*

This is a consequence of two facts: The Lie algebra generated by the drift and the vector field supporting the control generate the whole tangent space at each point (which entails some local controllability), and the uncontrolled flow (or *drift*) of the Kepler problem is recurrent. Under some additional convexity and compactness assumptions, one is then able to retrieve existence of optimal controls.

We now deal with such optimal controls. We restrict ourselves to integral cost functions, that is to problems of the form

$$\begin{cases} \dot{x}(t) = f(x(t), u(t)), \\ x(0) = x_0, \quad x(t_f) = x_f, \\ \displaystyle\int_0^{t_f} L(x(t), u(t))\,\mathrm{d}t \to \min \end{cases} \tag{4}$$

where the final time t_f can be fixed or not, and $L : M \times U \to \mathbb{R}$ is a smooth function. In the early 60s, Pontryagin and his coauthors realized that necessary conditions for optimality could be stated in Hamiltonian terms. By T^*M we denote the cotangent bundle of the manifold M.

Definition 1. The associated pseudo-Hamiltonian is

$$H : T^*M \times \mathbb{R} \times U \to \mathbb{R}, \quad (x, p, p^0, u) \mapsto \langle p, f(x, u)\rangle + p^0 L(x, u).$$

The following fundamental result is Pontryagin Maximum Principle [12] (see [11] for a modern presentation).

Theorem 1. *If (x, u) solves (4), there exist a Lipschitzian function $p(t) \in T^*_{x(t)}M$, $t \in [0, t_f]$, a constant $p^0 \leq 0$, $(p(t), p^0) \neq 0$, such that, almost everywhere,*

(i) *(x, p) is a solution of the Hamiltonian system associated with $H(\cdot, \cdot, u(t))$:*

$$\dot{x} = \frac{\partial H}{\partial p}(x, p, u), \quad \dot{p} = -\frac{\partial H}{\partial x}(x, p, u),$$

(ii) *$H(x(t), p(t), u(t)) = \max_{v \in U} H(x(t), p(t), v)$.*

Such curves (x, p) are called extremals. As a consequence of the maximization condition, the pseudo-Hamiltonian evaluated along an extremal is constant. Moreover, if the final time is free then this constant is zero.

This powerful result has some downsides. The Hamiltonian is defined on the cotangent bundle of the original phase space, and thus the dimension is doubled. Besides, the maximization condition, which "eliminates the control" and allows to obtain a truly Hamiltonian system in (x, p) only, might generate singularities (that is non-differentiability points of the maximized Hamiltonian which is in general only Lipschitzian as a function of time when evaluated along an extremal). The above theorem applies to time minimization with $L \equiv 1$ (and free final time). In this case, the non-positive constant p^0 is only related to the level of the Hamiltonian, and we will not mention it in the sequel as we will not discuss the implications of having *normal* ($p^0 \neq 0$) or *abnormal* ($p^0 = 0$) extremals.

2.2. Main result

The minimum time Kepler problem can be stated according to

$$\begin{cases} \ddot{q} + \dfrac{q}{\|q\|^3} = u, \quad \|u\| \leq 1, \\ (q(0), \dot{q}(0)) = (q_0, v_0), \quad (q(t_f), \dot{q}(t_f)) = (q_f, v_f), \\ t_f \to \min, \end{cases} \tag{5}$$

where, as before, $q \in \mathbb{R}^2$ is the position vector and $u \in \mathbb{R}^2$ the control. It will be convenient to use the same notations as in the general problem (4) and let

$$q = (x_1, x_2), \quad \dot{q} = (x_3, x_4),$$

be the coordinates on the initial phase space $M = (\mathbb{R}^2 \setminus \{0\}) \times \mathbb{R}^2$. According to Definition 1, the pseudo-Hamiltonian is then

$$H(x, p, u) = p_1 x_3 + p_2 x_4 - \frac{p_3 x_1 + p_4 x_2}{(x_1^2 + x_2^2)^{3/2}} + p_3 u_1 + p_4 u_2. \tag{6}$$

According to Theorem 1, minimizing trajectories must be projections on M of integral curves of the Hamiltonian that has to be maximized over the unit disk. The maximized Hamiltonian is readily equal to

$$H(x, p) = p_1 x_3 + p_2 x_4 - \frac{p_3 x_1 + p_4 x_2}{(x_1^2 + x_2^2)^{3/2}} + \sqrt{p_3^2 + p_4^2}$$

on T^*M, while the control is given by

$$u = \frac{1}{\sqrt{p_3^2 + p_4^2}}(p_3, p_4)$$

whenever p_3 and p_4 do not vanish simultaneously. Now, let

$$\mathcal{M} = \{(x, p, r) \in \mathbb{C}^8 \times \mathbb{C}^2_*, \ r_1^2 = x_1^2 + x_2^2, \ r_2^2 = p_3^2 + p_4^2\}$$

be the Riemann surface of H. It is a complex symplectic manifold (with local Darboux coordinates (x, p) outside the singular hypersurface $r_1 r_2 = 0$), over which H extends meromorphically, and even rationally, since

$$H(x, p, r) = p_1 x_3 + p_2 x_4 - \frac{p_3 x_1 + p_4 x_2}{r_1^3} + r_2. \tag{7}$$

The Hamiltonian H has four degrees of freedom, hence (see [1]) the meromorphic Liouville integrability of H over $\mathcal{M}$ would mean that there would exist three independent first integrals, in addition to H itself, almost everywhere in $\mathcal{M}$. The aim of this paper is to prove that it is not the case.

Theorem 2. *The minimum time Kepler problem is not meromorphically Liouville integrable on $\mathcal{M}$.*

It is well known that the classical Kepler problem is integrable, and even super integrable (since there are more first integrals than degrees of freedom, as a result of Kepler's first law and of the dynamical degeneracy of the Newtonian potential—see for instance [13]). On the opposite, the three-body problem is not as is known after the seminal work of Poincaré (for recent accounts on this topic see, *e.g.*, [14–17]). Similarly, the above theorem asserts that lifting the Kepler problem to the cotangent bundle and introducing the singular control term r_2 breaks integrability.

This result prevents the existence of enough complex analytic (and even meromorphic) first integrals to ensure integrability over $\mathcal{M}$. Of course, it does not prevent the existence of an additional real first integral which would have a natural frontier asymptotic to the real domain and thus, would not extend to the complex plane. Future work might be dedicated to investigate either or not Theorem 2 holds for real first integrals.

3. Proof of Theorem 2

The rest of the article is devoted to proving the theorem. Our proof consists in studying the variational equation along some integral curve of (7). In order to carry out this computation, we choose a collision orbit, with the drawback that it requires some regularization. We also note that there exist effective tools to perform this kind of computations (see, *e.g.*, [18]). The algebraic obstruction to Liouville integrability comes from the theorem below of Moralès and Ramis, which we now recall. We follow the presentation of Singer in [19].

3.1. Some facts of galois differential theory

Consider a linear differential equation $(L) : Y' = AY, A \in M_n(k)$, k being a differential field whose field of constants k_0 is algebraically closed, and of characteristic zero. We want the Galois group to be the group of symmetries preserving all algebraic and differential relations of a basis of solutions. We consider the polynomial ring

$$S = k[Y_{1,1}, \dots, Y_{n,n}, 1/\det(Y)]$$

where Y is an $n \times n$ matrix. This ring has a derivation provided by the differential system $Y' = AY$. We now consider a maximal differential ideal M of S, and the quotient $R = S/M$. This quotient satisfies the following

Definition 2 (*Picard–Vessiot Field*)**.** A Picard–Vessiot ring for $Y' = AY$ is a differential ring R over k such that

(i) The only differential ideals of R are (0) and R.
(ii) There exists a fundamental matrix $Z \in GL_n(R)$ for the equation $Y' = AY$.
(iii) R is generated as a ring by k, the entries of Z and $1/\det(Z)$.

It turns out that the choice of the maximal differential ideal M always gives the same Picard–Vessiot ring up to isomorphism. This ring is also a domain, thus allowing to consider the quotient field, the Picard–Vessiot field.

Definition 3 (*Galois Group*)**.** The differential Galois group of R over k is the group of differential automorphism of R preserving k, noted $Gal(R/k)$.

For a differential system $Y' = AY$, if there is no ambiguity on the base field k. (For the case treated in this paper, the base field k is $\mathbb{C}(z)$.) Given a fundamental matrix of solution Z and a Galois group element σ, we have $Z' = AZ$, and thus applying σ, we also have $\sigma(Z)' = A\sigma(Z)$. Thus $\sigma(Z)$ is also a matrix of solutions; there exists a constant matrix C such that $\sigma(Z) = ZC$, and as σ is an automorphism, C has to be invertible. So $Gal(R/k)$ can be represented as a group of $n \times n$ matrices.

Proposition 2. *The Galois group $Gal(R/k) \subset GL_n(k_0)$ is a linear algebraic group, i.e. the zero set in $GL_n(k_0)$ of a system of polynomials over k_0 in n^2 variables.*

Proof. This can be obtained by letting a Galois group element σ act (right multiplication by a matrix) on the differential ideal $I = (f_1, \dots, f_p)$. We can moreover assume that $f_i \in k[Y]$. As this does not change the degrees in the $Y_{i,j}$ and since I must be stabilized, $\sigma(f_i)$ must belong to $I \cap k_{\max(\deg f_1, \dots, \deg f_p)}[Y]$. This condition is a condition of membership to a vector space, which provides algebraic conditions on the entries of the matrix σ.

Proposition 3 (*Fundamental Theorem of Differential Galois Theory*)**.** *Let K be a Picard–Vessiot field with differential Galois group G over k.*

(i) *There is a one-to-one correspondence between Zariski-closed subgroups $H \subset G$ and differential subfields F, $k \subset F \subset K$, given by*

$$H \subset G \to K^H = \{a \in K, \sigma(a) = a \ \forall \sigma \in H\}$$

$$F \to Gal(K/F) = \{\sigma \in G, \sigma(a) = a \ \forall a \in F\}$$

(ii) *A differential subfield F, $k \subset F \subset K$, is a Picard–Vessiot extension of k if and only if $Gal(K/F)$ is a normal subgroup of G, in which case $Gal(F/k) \simeq G/Gal(K/F)$.*

We are interested in non integrability for nonlinear Hamiltonian systems. The link with the Hamiltonian world is given by the celebrated theorem of Moralès–Ramis below. We recall that an algebraic group G is said to be virtually Abelian if its connected component containing the identity is an Abelian subgroup of G.

Theorem 3 (*Moralès–Ramis [8]*)**.** *Let H be an analytic Hamiltonian on a complex analytic symplectic manifold and Γ be a non constant solution. If H is integrable in the Liouville sense with meromorphic first integrals, then the first order variational equation along Γ has a virtually Abelian Galois group over the base field of meromorphic functions on Γ.*

The main idea behind this theorem is that if H is Liouville integrable, then so are the linearized equations near a non constant solution Γ. More precisely, thanks to Ziglin's lemma below, the first integrals of H can be transformed in such a way that their first non trivial term in their series expansion near Γ is functionally independent.

Lemma 1 (*Ziglin's Lemma*)**.** *Let $\Phi_1, \ldots, \Phi_r \in k(x_1, \ldots, x_n)$ be functionally independent functions. We consider $\Phi_1^0, \ldots, \Phi_r^0$ the lowest degree homogeneous term for some fixed positive weight homogeneity in $x_1, \ldots, x_n$. Assume $\Phi_1^0, \ldots, \Phi_{r-1}^0$ are functionally independent. Then there exists a polynomial Ψ such that the lowest degree homogeneous term Ψ^0 of $\Psi(\Phi_1, \ldots, \Phi_r)$ is such that $\Phi_1^0, \ldots, \Phi_{r-1}^0, \Psi^0$ are functionally independent.*

Applying this lemma recursively, we prove that if a Hamiltonian system admits a set of commuting, functionally independent meromorphic first integrals on a neighborhood of a curve, then their first order terms, after possibly polynomial combinations of them, are also commuting, functionally independent meromorphic first integrals of the linearized system along the curve. Moralès–Ramis [8] precisely proved that symplectic linear differential systems having such first integrals have a Galois group whose identity component is Abelian. This result can be expected knowing that the Galois group leaves invariant every first integral, so the more first integrals, the smaller the Galois group.

We will need the definition of the monodromy group of a linear differential equation. Let us consider a differential system $Y' = AY$, $A \in M_n(\mathbb{C}(x))$. We note $S = \mathbb{P}^1 \backslash \{\text{singularities of } A\}$. Let us consider a point $z_0 \in S$ and a closed oriented curve $\gamma \subset S$, with $x_0 \in \gamma$. There exists a basis of solutions Z on a neigbourhood of x_0, holomorphic in z. We now use analytic continuation along the loop γ to extend this basis of solutions. However, it cannot *a priori* be extended to a whole neighborhood of γ, because after one loop, the basis of solutions Z_γ at x_0 could be different. This defines a matrix $D_\gamma \in GL_n(\mathbb{C})$ such that $Z_\gamma = ZD_\gamma$ and thus a homomorphism

$$\mathrm{Mon} : \pi_1(S, x_0) \to GL_n(\mathbb{C}), \quad \mathrm{Mon}(\gamma) = D_\gamma.$$

This homomorphism carries the group structure of $\pi_1(S, x_0)$, and thus its image is also a group.

Definition 4. The image of the application Mon is called the monodromy group.

Note that the monodromy group depends on the choice of Z, so it is only determined up to conjugation. Since analytic continuation preserves analytic relations, the monodromy group is a subset of the differential Galois group over the base field of meromorphic functions on S; in particular, it is included in the differential Galois group over the base field of rational functions. For Fuchsian systems (all singularities are regular singularities, i.e. the growth at singularities of solutions is at most polynomials), we have moreover the following.

Theorem 4 (*Schlesinger Density Theorem [20]*)**.** *Let (E) : $Y' = AY$ be a Fuchsian differential linear equation with coefficients in $\mathbb{C}(x)$ and let Π be its monodromy group. Then Π is dense for the Zariski topology in the Galois group of the Picard–Vessiot extension of (E) over the base field of rational functions: $\overline{\Pi} = Gal(A)$.*

3.2. A collision orbit

In order to find an explicit solution of (6), let us define the 4-dimensional symplectic submanifold

$$S = \{(x, p, r) \in \mathcal{M} \mid x_2 = x_4 = p_2 = p_4 = 0, r_1 = x_1, r_2 = -p_3\}.$$

As S is the phase space of the controlled Kepler problem on the line (collision orbit) parameterized by q_1, it is invariant. On the interior of S, (x_1, x_3, p_1, p_3) is a set of (Darboux) coordinates and, in restriction to S, the Hamiltonian reduces to

$$H(x, p) = p_1 x_3 - \frac{p_3}{x_1^2} - p_3,$$

so the Hamiltonian vector field on S is

$$\begin{cases} \dot{x}_1 = x_3 \\ \dot{x}_3 = -1 - \dfrac{1}{x_1^2} \\ \dot{p}_1 = -\dfrac{2p_3}{x_1^3} \\ \dot{p}_3 = -p_1. \end{cases}$$

In particular,

$$\begin{cases} \ddot{x}_1 = -1 - \dfrac{1}{{x_1}^2} \\ \ddot{p}_3 - \dfrac{2p_3}{x_1^3} = 0. \end{cases} \tag{8}$$

As is known since the work of Charlier and Saint Germain on the Kepler problem with a constant force (see [21]), the function

$$C = \frac{1}{2}x_3^2 + x_1 - \frac{1}{x_1}$$

is a first integral on S and $H_{|S}$ is integrable. Let us change time to $s = x_1(t)$ and denote by $' = \frac{d}{ds}$ the derivation with respect to this new time. It suffices to find an obstruction in this modified time, as explained at the end of the proof.

Using (8), we see that the variable p_3 satisfies the linear differential equation

$$2\left(C + \frac{1}{x_1} - x_1\right)p_3''(x_1) - \left(1 + \frac{1}{x_1^2}\right)p_3'(x_1) - \frac{2p_3(x_1)}{x_1^3} = 0,$$

which yields

$$p_3(x_1) = \frac{\sqrt{-Cx_1 + x_1^2 - 1}}{\sqrt{x_1}}\left(c_1 \int \frac{x_1^{3/2}}{(-Cx_1 + x_1^2 - 1)^{3/2}}dx_1 + c_2\right)$$

for some constants of integration c_1 and c_2. Here the symbol $\int f(x_1)dx_1$ denotes some primitive of f with respect to the variable x_1. It suffices to find one particular integral curve along which the variational equation has a non virtually Abelian Galois group. To this end, we consider the simple – but rich enough – case $c_1 = 0, c_2 = 1$.

$$p_3(x_1) = \frac{\sqrt{-Cx_1 + x_1^2 - 1}}{\sqrt{x_1}}.$$

Using the expression of the first integral C and of the vector field, we deduce

$$x_3(x_1) = \sqrt{2}\frac{\sqrt{-Cx_1 + x_1^2 - 1}}{\sqrt{x_1}}, \quad p_1(x_1) = -\frac{1}{\sqrt{2}}\frac{x_1^2 + 1}{x_1^2}.$$

Choosing $C = 2i$ and some determination of the squares yields a particularly simple solution Γ drawn on $S \subset \mathcal{M}$,

$$\begin{cases} x_1 = x_1, \\ x_2 = 0, \\ x_3 = \sqrt{2}\dfrac{x_1 - i}{\sqrt{x_1}}, \\ x_4 = 0, \end{cases} \qquad \begin{cases} p_1 = -\dfrac{x_1^2 + 1}{\sqrt{2}x_1^2}, \\ p_2 = 0, \\ p_3 = \dfrac{x_1 - i}{\sqrt{x_1}}, \\ p_4 = 0. \end{cases} \tag{9}$$

3.3. Normal variational equation

In the initial time, the linearized equation along Γ is the Hamiltonian vector field associated with the Hamiltonian DH along Γ:

$$\dot{Z}(t) = A(t)Z(t), \quad A(t) = J\,D^2H(\Gamma(t)),$$

where J is the Poisson structure. In the coordinates $(x_1, \ldots, x_4, p_1, \ldots, p_4)$,

$$J = \begin{pmatrix} 0_4 & I_4 \\ -I_4 & 0_4 \end{pmatrix}.$$

We will keep on using time x_1, instead of the initial time t, writing

$$Z'(x_1(t)) = \frac{1}{x_3(t)} A(x_1(t)) Z(x_1(t)).$$

Let us now reorder coordinates according to $(x_1, x_3, p_1, p_3, x_2, x_4, p_2, p_4)$. Since S is an invariant submanifold, the 8×8 matrix A has an upper triangular bloc structure

$$A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}$$

with

$$A_3 = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{\sqrt{2}p_3} \\ -\frac{1}{\sqrt{2}p_3^2} & 0 & -\frac{1}{\sqrt{2}x_1^3 p_3} & 0 \\ 0 & \frac{1}{\sqrt{2}p_3} & 0 & 0 \\ -\frac{1}{\sqrt{2}x_1^3 p_3} & 0 & \frac{3}{\sqrt{2}x_1^4} & 0 \end{pmatrix}.$$

Moralès–Ramis Theorem gives necessary conditions for Liouville integrability in terms of the Galois group of this linear differential system over the base field of meromorphic functions *on* Γ. Looking at the expression (9) of Γ, we see that meromorphic functions on Γ are just meromorphic functions in $\sqrt{x_1} \in \mathbb{C} \setminus \{0, \pm\sqrt{i}\}$. The block A_3 corresponds to infinitesimal variations in the normal direction to S, which is the part where interesting phenomena might occur. As the Picard–Vessiot field is generated by all the components of the solutions, the Picard–Vessiot field K generated by the normal variational equation

$$(L): \quad X' = A_3 X, \quad X = (X_1, X_2, X_3, X_4)$$

is a subfield of the Picard–Vessiot field of the whole variational equation, and thus $\mathrm{Gal}(A) \supset \mathrm{Gal}(A_3)$. That $\mathrm{Gal}(A_3)$ is not virtually Abelian will thus imply that $\mathrm{Gal}(A)$ itself is not virtually Abelian. In order to reduce the system to a one dimensional linear equation, we use the cyclic vector method on A_3: From (L) we get $X_1' = L_1(X_1, X_2, X_3, X_4)$, where L_1 is a linear form on $\mathbb{R}^4$, thus by derivation,

$$\begin{aligned} X_1'' &= L_1(X_1', X_2, X_3, X_4) + L_1(X_1, X_2', X_3, X_4) \\ &\quad + L_1(X_1, X_2, X_3', X_4) + L_1(X_1, X_2, X_3, X_4') \\ &= L_2(X_1, X_2, X_3, X_4). \end{aligned}$$

Iterating, we obtain

$$\begin{cases} X_1 &= X_1, \\ X_1' &= L_1(X_1, X_2, X_3, X_4), \\ X_1'' &= L_2(X_1, X_2, X_3, X_4), \\ X_1^{(3)} &= L_3(X_1, X_2, X_3, X_4), \\ X_1^{(4)} &= L_4(X_1, X_2, X_3, X_4). \end{cases}$$

The L_i's are five linear forms on $\mathbb{R}^4$, so X_1 must satisfy some linear differential equation of order 4 that we compute to be

$$\begin{aligned} &X_1^{(4)} + \frac{2(3i - 5x_1)}{x_1(i - x_1)} X_1^{(3)} + \frac{(-3x_1 + i)(-29x_1 + 23i)}{4(x_1 - i)^2 x_1^2} X_1'' \\ &- \frac{(i - 3x_1)(7x_1 + i)}{4(x_1 - i)^2 x_1^3} X_1' + \frac{3x_1 + i}{4(x_1 - 1)^3 x_1^4} X_1 = 0. \end{aligned} \tag{10}$$

We find a solution of this equation of the form

$$X_1(x_1) = \frac{i - x_1}{\sqrt{x_1}} \left(c_1 + c_2 \int \sqrt{x_1} (1 + ix_1)^{-\frac{3}{2} - i\frac{\sqrt{3}}{2}} . {}_2F_1(\gamma(x_1)) dx_1 \right),$$

where ${}_2F_1$ is the Gauss hypergeometric function and

$$\gamma(x_1) = \left(\frac{5}{2} - i\frac{\sqrt{3}}{2}, \frac{1}{2} + i\frac{\sqrt{3}}{2}, 1 + i\sqrt{3}, 1 + ix_1 \right).$$

The Picard–Vessiot field K contains this solution and, as it is a differential field, it also contains

$$\sqrt{x_1} (1 + ix_1)^{-\frac{3}{2} - i\frac{\sqrt{3}}{2}} {}_2F_1(\gamma(x_1)).$$

Noting $\tilde{K}$ the differential field generated by this function, we have $\tilde{K} \subset K$. Now the Galois group of ${}_2F_1(\gamma(x_1))$ over $\mathbb{C}(x_1)$ is $SL_2(\mathbb{C})$ (see Kimura's table, [22]). By Galois correspondence, the Galois group of (10) over the rational functions in x_1 admits $SL_2(\mathbb{C})$ as a subgroup. The hypergeometric equation (10) is Fuchsian (all its singular points are regular), so thanks to Theorem 4, we know that its Galois group over the field of rational functions is the closure of its monodromy group. Besides, the Galois group over meromorphic functions contains the monodromy group, and of course, is included in the Galois group over rational functions. Eventually, the Galois group of (10) over meromorphic functions in x_1 also contains $SL_2(\mathbb{C})$. Thus, adding the algebraic extension $\sqrt{x_1}$, the Galois group can be reduced to at most one subgroup of index 2: The only possibility is that the identity component contains $SL_2(\mathbb{C})$ again. So the Galois group of K over the base field of meromorphic functions in $\sqrt{x_1} \in \mathbb{C} \setminus \{0, \pm\sqrt{i}\}$ contains $SL_2(\mathbb{C})$ and is not virtually Abelian. According to Moralès–Ramis, this concludes the proof.

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Further reading

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