JOURNAL OF OPTIMIZATION THEORY AND APPLICATIONS: Vol. 118, No. 3, pp. 541–565, September 2003 (© 2003)

3D Geosynchronous Transfer of a Satellite: Continuation on the Thrust[1]

J. B. Caillau,[2] J. Gergaud,[3] and J. Noailles[4]

Communicated by H. J. Oberle

Abstract. The minimum-time transfer of a satellite from a low and eccentric initial orbit toward a high geostationary orbit is considered. This study is preliminary to the analysis of similar transfer cases with more complicated performance indexes (maximization of payload, for instance). The orbital inclination of the spacecraft is taken into account (3D model), and the thrust available is assumed to be very small (e.g. 0.3 Newton for an initial mass of 1500 kg). For this reason, many revolutions are required to achieve the transfer and the problem becomes very oscillatory. In order to solve it numerically, an optimal control model is investigated and a homotopic procedure is introduced, namely continuation on the maximum modulus of the thrust: the solution for a given thrust is used to initiate the solution for a lower thrust. Continuous dependence of the value function on the essential bound of the control is first studied. Then, in the framework of parametric optimal control, the question of differentiability of the transfer time with respect to the thrust is addressed: under specific assumptions, the derivative of the value function is given in closed form as a first step toward a better understanding of the relation between the minimum transfer time and the maximum thrust. Numerical results obtained by coupling the continuation technique with a single–shooting procedure are detailed.

Key Words. Low thrust orbit transfer, minimum-time control, continuation technique, parametric control, shooting method.

[1]This research was supported by the French Ministry for Education, Higher Education and Research, Grant 20INP96, and the French Space Agency, Contract 86/776/98/CNES/7462. An extended abstract of this paper in the coplanar case was presented at the Conference on Nonlinear Analyis 2000, Courant Institute, New York, NY (Ref. 1).

[2]Assistant Professor, Applied Mathematics and Computer Science Department of ENSEEIHT, CNRS and Institut National Polytechnique, Toulouse, France.

[3]Assistant Professor, Applied Mathematics and Computer Science Department of ENSEEIHT, CNRS and Institut National Polytechnique, Toulouse, France.

[4]Professor, Applied Mathematics and Computer Science Department of ENSEEIHT, CNRS and Institut National Polytechnique, Toulouse, France.

1. Introduction

We are interested in the application of optimal control to spaceflight dynamics. More precisely, we consider the minimum-time transfer of a satellite in the gravitation field of the Earth. Contrary to the case of aeroassisted orbital transfers (see Ref. 2), we start from a low initial orbit and aim at reaching a high geosynchronous orbit (nonatmospheric transfer). As in Ref. 3, the transfer time is minimized and both the eccentricity and the inclination have to be corrected (3D transfer; see Refs. 4–5 for the coplanar case). In the context of electro-ionic propulsion (Refs. 6–7), the maximum thrust of the engine is very small: while in Ref. 2 the thrust varies from 1270000 Newtons (impulsive control) to 5000 Newtons for an initial mass of 15000 kg, we shall work with thrusts between 60 Newtons and 0.14 Newton with a satellite of 1500 kg (that is, with an acceleration 2000 times smaller in the lower case). As a result, the transfer times are very long (up to several months) and the numerical solution of the problem requires specific techniques. A natural idea is then to connect the simple problems (with strong thrusts) to the difficult ones (with low thrusts): this well-known procedure is often referred to as homotopy or continuation (Refs. 8–9). Actually, as we shall see in Section 3, it turns out that this continuation on the maximum thrust is all the more relevant here that the dependence of the transfer time on the thrust proves to be quite smooth.

We formulate first the transfer as an optimal control problem in Section 2 and give some preliminary results. The continuation process is studied in Section 3: we prove that, under reasonable assumptions, the value function of a general optimal control problem is right-continuous with respect to the essential bound on the control. In the transfer case, the specific structure of the problem allows even a C^1-sensitivity analysis, and the derivative of the transfer time as a function of the maximum thrust can be explicited in Section 4 under assumptions typical of parametric control (Refs. 10–11). The numerical computation of the optimal trajectories using single shooting(see Ref. 12 for a direct transcription approach on similar problems) coupled with continuation on the bound on the control is described in Section 5. The results are given for the 3D model and very low thrusts, down to 0.14 Newton.

2. Statement of the Problem and Preliminary Results

In order to give the mathematical formulation of the transfer problem, we suppose that the satellite can be modeled as a mass point and we neglect the high-order terms of the Earth gravitational field (we just consider an

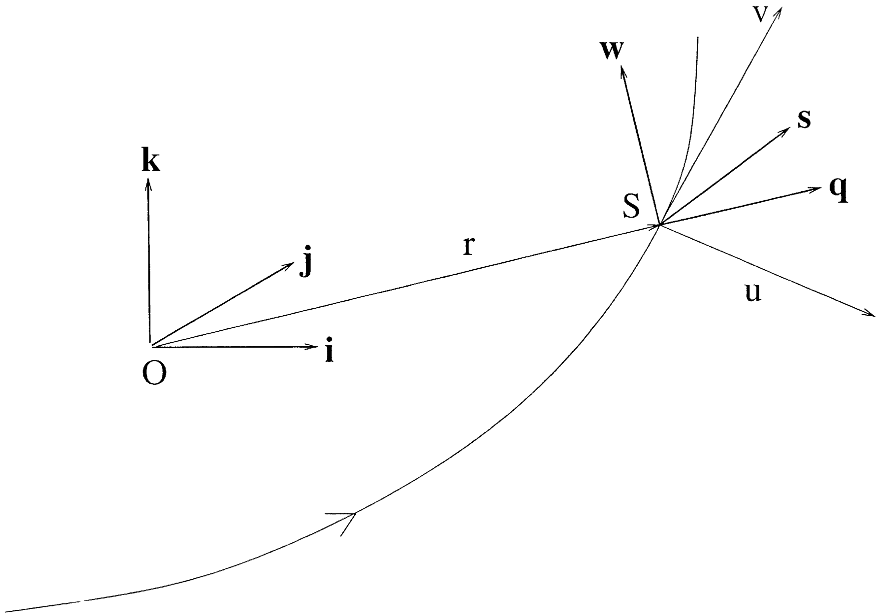

Fig. 1. Local frame (q, s, w) for the thrust; (r, v) is the rain position-speed of the satellite S.

inverse square law field). Hence, in the so-called Cartesian coordinates, $r=(r_1, r_2, r_3)$ being the position vector and $u=(u_1, u_2, u_3)$ the thrust of the engine, the dynamics is

$$\ddot{r} = -\mu^0 r/|r|^3 + u/m. \tag{1}$$

In (1), $\mu^0 = \mathscr{G} m_T \simeq 398600.47\,\mathrm{km}^3\,\mathrm{s}^{-2}$ is the Earth gravitation constant,[5] m the mass of the satellite, and $|\cdot|$ the Euclidean norm, $|r| = (r_1^2 + r_2^2 + r_3^2)^{1/2}$; the same convention will be used for all finite-dimensional norms in the rest of the paper. But rather than using the position-speed variables, we prefer to use the Gauss coordinate's that describe the ellipse osculating to the trajectory (Ref. 13). In these coordinates, the dynamics is still affine in the control, which is expressed in a local frame attached to the ellipse [reference (q, s, w), see Fig. 1]:

$$\dot{x} = f_0(x) + (1/m) \sum_{i=1}^{3} u_i f_i(x). \tag{2}$$

[5] $\mathscr{G}$ is the universal gravitation constant and m_T the mass of the Earth.

The state is $x=(P, e_x, e_y, h_x, h_y, L)$, where the variables define the geometry of the osculating ellipse and the position of the satellite on at: P is the semilatus rectum; (e_x, e_y) is the eccentricity vector; (h_x, h_y) is the inclination vector; L is the true longitude. These coordinates are well suited, since all but L are first integrals of the free motion (i.e., without control): hence, contrary to the Cartesian coordinates, they vary slowly for low thrust transfers [in the 2D case, the lowest thrust solvable by shooting when using Cartesian coordinates is only 0.7 Newton (see Ref. 14)]. The vector fields in (2) are

$$f_0=\sqrt{\mu^0/P}\begin{bmatrix}0\\0\\0\\0\\0\\W^2/P\end{bmatrix}, \tag{3}$$

$$f_1=\sqrt{P/\mu^0}\begin{bmatrix}0\\\sin L\\-\cos L\\0\\0\\0\end{bmatrix}, \tag{4}$$

$$f_2=\sqrt{P/\mu^0}\begin{bmatrix}2P/W\\\cos L+(e_x+\cos L)/W\\\sin L+(e_y+\sin L)/W\\0\\0\\0\end{bmatrix}, \tag{5}$$

$$f_3=1/W\ \sqrt{P/\mu^0}\begin{bmatrix}0\\-Ze_y\\Ze_x\\(C/2)\cos L\\(C/2)\sin L\\Z\end{bmatrix}, \tag{6}$$

with

$$W = 1 + e_x \cos L + e_y \sin L,$$
$$Z = h_x \sin L - h_y \cos L,$$
$$C = 1 + h_x^2 + h_y^2.$$

Accordingly, the dynamics is well defined and smooth[6] on the following open (smooth) submanifold of $\mathbb{R}^6$:

$$M = \{x \in \mathbb{R}^6 | P > 0, e_x^2 + e_y^2 < 1\}.$$

Note that, with

$$e_x^2 + e_y^2 < 1,$$

we restrict ourselves to elliptic trajectories. We also take into account the fact that the mass flow is proportional to the modulus of the thrust,

$$\dot{m} = -\beta |u|, \tag{7}$$

so that the state of the satellite is in fact $(x, m) \in M \times \mathbb{R}_+^*$. To ensure compactness of the set of admissible trajectories, we restrict ourselves to a security zone $A \subset \mathbb{R} \times M$ of the tx-plane defined by

$$t \geq 0, \qquad P \geq \Pi^0, \tag{8}$$

with $\Pi^0 > 0$ (the path constraint $P \geq \Pi^0$ prevents the satellite from colliding with the Earth). Analogously, the mass of the satellite has to remain greater than the mass without fuel χ^0,

$$m \geq \chi^0. \tag{9}$$

The initial and the terminal orbits are prescribed, so we have the boundary constraints

$$x(0) = x^0, \qquad m(0) = m^0, \qquad h(x(t_f)) = 0, \tag{10}$$

with

$$x^0 = (P^0, e_x^0, e_y^0, h_x^0, h_y^0, L^0) \in \mathbb{R}^6,$$
$$h(x) = (P - P^f, e_x - e_x^f, e_y - e_y^f, h_x - h_x^f, h_y - h_y^f) \in \mathbb{R}^5,$$

and

$$P^0 = 11625 \text{ km}, \qquad P^f = 42165 \text{ km}, \tag{11a}$$
$$e_x^0 = 0.75, \qquad e_x^f = 0, \tag{11b}$$

[6]Smooth stands for C^∞–smooth.

$$e_y^0 = 0, \qquad e_y^f = 0, \tag{11c}$$

$$h_x^0 = 0.0612, \qquad h_x^f = 0, \tag{11d}$$

$$h_y^0 = 0, \qquad h_y^f = 0, \tag{11e}$$

$$L^0 = \pi, \qquad L^f \text{ free}, \tag{11f}$$

$$m^0 = 1500 \text{ kg}, \qquad h^f \text{ free}. \tag{11g}$$

According to (11), the initial orbit is very eccentric [one has $|(e_x^0, e_y^0)| = 0.75$]. On the contrary, the inclination with respect to the equatorial plane is weak, since $|(h_x^0, h_y^0)| = 0.0612$ corresponds to an angle of 7 degrees. Here, the final longitude is free, so we do have an orbit transfer problem, not a rendezvous problem.[7] Finally, we consider the constraint on the control,

$$|u| \leq T_{\max}, \tag{12}$$

meaning that the thrust is limited in intensity by $T_{\max}$. As before, (12) is equivalent to

$$u_1^2 + u_2^2 + u_3^2 \leq T_{\max}^2.$$

Our transfer problem, parameterized by the maximum thrust $T_{\max}$, is thus to find an absolutely continuous state (x, m) in $W_7^{1,\infty}([0, t_f]) = W^{1,\infty}([0, t_f], \mathbb{R}^7)$ and an essentially bounded control u in the space $L_3^\infty([0, t_f]) = L^\infty([0, t_f], \mathbb{R}^3)$ that minimize the transfer time t_f,

$$\min t_f,$$

and that match the dynamics (2), (7), the path constraints (8)–(9), the boundary constraints (10), and the control constraint (12). For a given $T_{\max}$, the problem will be referred to as $(\mathrm{SP})_{T_{\max}}$.

It is proven in Ref. 5 that, no matter how low the thrust might be, the system remains controllable provided the mass of the satellite without fuel χ^0 is small enough. This property comes from the fact that the Lie algebra defined by the vector fields (3)–(6) has maximal rank and that the drift [that is, the Keplerian action modeled by f_0] is periodic. Hence, the set of admissible trajectories and controls is nonempty and the existence of optimal controls proceeds from the Filippov theorem (Ref. 15). Indeed, the dynamics is clearly convex in the control; the control set $U = B_c(0, T_{\max})$ is also convex; $B_c(a, \rho)$ denotes the closed Euclidean ball with center a and (strictly positive) radius ρ; and though the *txm*–space is unbounded, one can construct a Lyapunov function to show that x and m remain in a fixed compact subset

[7] In (11), the initial longitude is assumed to be prescribed for convenience.

of $M \times \mathbb{R}$ (moreover, since the criterion is t_f and since there are admissible trajectories, one can always assume that t_f is smaller than a given T, with T big enough).

Now, consider the following assumption:

(A1) Any optimal trajectory (including the mass) is interior to the path constraints (8)–(9).

Under (A1), the necessary condition for optimality (Refs. 15–16) applies and, if (t_f, x, m, u) in $\mathbb{R} \times W_7^{1,\infty}([0, t_f]) \times L_3^\infty([0, t_f])$ is optimal, there exists an absolutely continuous adjoint state p such that the control verifies

$$u = -T_{\max}\psi/|\psi|, \tag{13}$$

whenever ψ does not vanish, where $\psi(t) = B(x(t))^T p(t)$ is the so-called switching function of the problem [$B(x)$ is the 6×3 matrix $[f_1, f_2, f_3]$, and T denotes the transpose operator]. Indeed, it is proven in Ref. 5 that, although we take into account the mass variation [Eq. (7)], the control is almost everywhere of maximum modulus.[8] Furthermore, the geometric analysis implies that there is only a finite number of switching points. More precisely, consider the next assumption:

(A2) the constraints of $(\mathrm{SP})_{T_{\max}}$ are qualified.

Under (A2), we are able to give precise bounds on the number of consecutive switchings located at the perigee of the osculating ellipse. As explained in Ref. 5, this is the justification for the practical assumption that there is no switching at all, namely:

(A3) any optimal control is continuous.

Remark 2.1. By the Pontryagin maximum principle,[9] $|u| = T_{\max}$ almost everywhere [that is, everywhere, with (A3)]. Hence, the mass is known explicitly as a function of the time,

$$m(t) = m^0 - t\beta T_{\max}, \tag{14}$$

and $(\mathrm{SP})_{T_{\max}}$ can be given an equivalent nonautonomous formulation. However, we will sometimes need to consider the original formulation where, contrary to (14), the parameter $T_{\max}$ does not appear in the dynamics (e.g., at the end of Section 3).

Before going to the next section, devoted to the continuation process, we finish by connecting the 3D model to the 2D model previously studied

[8]The coplanar study extends straightforwardly to the 3D case; see Ref. 14.

[9]Though we minimize the Hamiltonian, we still refer to the usual first-order necessary condition as the Pontryagin maximum principle.

in Refs. 4–5. The latter is not an approximation of the former but truly a particular case (in this sense, it is still an exact 3D model). Indeed, it is obtained by adding the path constraint that the inclination is constant, that is,

$$|(h_x, h_y)| = \text{const.} \tag{15}$$

According to (15), one then has

$$h_x \dot{h}_x + h_y \dot{h}_y = 0,$$

and since the trajectory has also to verify the dynamics,

$$|(h_x, h_y)| \sqrt{P/\mu^0}(1 + |(h_x, h_y)|^2)\, u_3 \cos(L - \Omega)/2\, W = 0, \tag{16}$$

where Ω is the so-called longitude of the ascending node [that is, $(h_x, h_y) = |(h_x, h_y)|(\cos\Omega, \sin\Omega)$, see Ref. 13]. In all cases, (16) implies that u_3 is identically zero (no offplane component of the thrust), in order that the variables h_x and h_y can be eliminated from the dynamics (since they appear only in the vector field f_3).

A remarkable feature of the 2D optimal trajectories is that they remain extremal for the 3D model whenever the inclination has to be the same at the initial and final times.

Proposition 2.1. Under the additional boundary constraint that the inclination is the same at $t = 0$ and $t = t_f$, the 2D optimal trajectories are extremals of the 3D model.

Proof. Let us consider the constraint $(h_x, h_y)(0) = (h_x, h_y)(t_f)$, same inclination at the times 0 and t_f. If $\bar{u} = (\bar{u}_1, \bar{u}_2)$ is an optimal control of the 2D problem, let $u = (\bar{u}_1, \bar{u}_2, 0)$; u remains admissible thanks to the linearity of the dynamics in the control and to the fact that neither u_1 nor u_2 appear in the equations for $\dot{h}_x$ and $\dot{h}_y$ [see Eqs. (3)–(6)] in order that h is constant (thus verifying the new boundary constraint). Besides, since the vector fields f_0, f_1, f_2 do not depend on (h_x, h_y), the associated adjoint equations are written as

$$\dot{p}_{h_x} = -u_3/m \partial_{h_x} f_3^T p, \tag{17}$$

$$\dot{p}_{h_y} = -u_3/m \partial_{h_y} f_3^T p. \tag{18}$$

Hence, it is enough to choose p_{h_x} and p_{h_y} identically zero to satisfy (13), (17, 18), as well as the transversality condition

$$(p_{h_x}, p_{h_y})(0) = (p_{h_x}, p_{h_y})(t_f). \qquad \square$$

At this point, it is not yet clear whether an optimal 2D trajectory is, more than extremal, optimal under the previous additional boundary constraint on the inclination. Indeed, one can imagine that there are better strategies that use an offplane thrust during the transfer. Nevertheless, the extremality given by Proposition 2.1 is important, since in practice the control problem is solved by means of indirect methods (e.g. single shooting; see Section 5) that actually find the extremals.

3. Continuation on the Essential Bound

We consider in this section the following general optimal control problem: find a real t_f (the final time is supposed to be free[10]), an absolutely continuous state x in $W_n^{t,\infty}([0, t_f])$, and an essentially bounded control u in $\mathrm{L}_m^{\infty}([0, t_f])$ such that the criterion below is minimized (Mayer form),

$$\min g(t_f, x(t_f)). \tag{19}$$

The dynamics is smooth on an open submanifold M of $\mathbb{R}^n$,

$$\dot{x} = f(t, x, u), \qquad t \in [0, t_f], \tag{20}$$

and we consider the boundary constraints

$$x(0) = x^0, \qquad h(x(t_f)) = 0, \tag{21}$$

where h is a smooth submersion of $\mathbb{R} \times M$ on $\mathbb{R}^l$, together with the state and control constraints

$$(t, x) \in A, \qquad u \in U_\rho(t, x), \tag{22}$$

where A is a closed subset of $\mathbb{R} \times M$. The parameter ρ acts in (22) according to

$$U_\rho(t, x) = U(t, x) \cap B_c(0, \rho),$$

with as before $B_c(0, \rho)$ the closed Euclidean ball of radius $\rho > 0$ centered at the origin. The parametric control problem (19)–(22) will be referred to as $(\mathrm{OCP})_\rho$. The control constraint $u \in U_\rho(t, x)$ is a priori all the more difficult as ρ is smaller (and so is the numerical computation). For this reason, we consider homotopy on the parameter ρ, so as to connect the complicated problems with ρ small to simpler ones with ρ big. What we do practically (see Section 5) is discrete continuation; that is, we use a decreasing sequence of positive values $(\rho_k)_k$ in order to generate a sequence of optima. Such

[10]The results given here remain valid for problems with fixed final time with obvious modifications.

techniques are commonly used in numerical optimal control (see e.g. Refs. 8–9). The minimal property required to justify this kind of process in our case is that, if $(\rho_k)_k$ decreases toward $\rho > 0$, the sequence of optimum values tends to the optimum value of the limit problem. Namely, what we need is the right-continuity of the value function V that maps the parameter ρ to the optimum value $V(\rho) \in \overline{\mathbb{R}}$ of $(\mathrm{OCP})_\rho$. We give sufficient conditions that entail this regularity. These assumptions will be immediately verified in the minimum-time transfer case.

(H1) The set of admissible triples (t_f, x, u) for $(OCP)_\rho$ is nonempty for any $\rho > 0$.

(H2) $N_\rho = \{(t, x, u) \in \mathbb{R} \times M \times \mathbb{R}^m \,|\, (t, x) \in A, u \in U_\rho(t, x)\}$ is compact.

(H3) $Q_\rho(t, x) = f(t, x, U_\rho(t, x))$ is convex for any $(t, x) \in \mathbb{R} \times M$ and $\rho > 0$.

These conditions are nothing else but the classical sufficient conditions for existence so that obviously we have the following proposition.

Proposition 3.1. Under Assumptions (H1)–(H3), the value function V is finite and decreasing.

Proof. $V(\rho)$ is finite for any (strictly) positive ρ by the Filippov theorem. Besides, if $0 < \rho_1 \leq \rho_2$, the set of admissible triples for $(\mathrm{OCP})_{\rho_1}$ is clearly included in the one for $(\mathrm{OCP})_{\rho_2}$ in order that $V(\rho_2) \leq V(\rho_1)$. □

As a consequence, V which is monotonous has only a countable number of discontinuities. To ensure the right-continuity, we suppose that the dynamics can be smoothly inverted:

(H4) There are smooth functions R and S, $R(t, x)$ in $\mathscr{L}(\mathbb{R}^n, \mathbb{R}^m)$, $S(t, x)$ in $\mathbb{R}^m$, such that, if $y = f(t, x, u)$, then $u = R(t, x)y + S(t, x)$.

This requirement is fulfilled, e.g., as soon as the dynamics is affine in the control,

$$\dot{x} = f_0(t, x) + B(t, x)\, u,$$

with $B(t, x)$ an embedding. Then, we have the following proposition.

Proposition 3.2. Under Assumptions (H1)–(H4), the value function V is right-continuous on $\mathbf{R}_+^*$.

We shall use the following fact for the proof. Let $(y_k)_k$ be a bounded sequence in $\mathrm{L}_n^\infty([0, T])$, converging to $y \in \mathrm{L}_n^\infty([0, T])$ in the space of Schwartz distributions (see Ref. 17): for any compactly supported smooth (vector-valued) function ϕ on [0, T], one has

$$\int_{[0,T]} (y_k|\phi)\,dt \to \int_{[0,T]} (y|\phi)\,dt, \tag{23}$$

with $(\cdot|\cdot)$ the Euclidean dot product. Then, $(y_k)_k$ is equicontinuous in $\mathrm{L}_n^\infty([0, T])$ identified with the dual space of $\mathrm{L}_n^1([0, T])$ and weakly$*$ converging toward y: (23) still holds for all ϕ in $\mathrm{L}_n^1([0, T])$.

Proof of Proposition 3.2. Let $(\rho_k)_k$ be a sequence decreasing toward $\rho > 0$. We denote by (t_{f_k}, x_k, u_k) a solution of $(\mathrm{OCP})_k$. Since A is compact by (H2), there is a positive T such that $A \subset [0, T] \times \mathbb{R}^n$ and we extend each x_k [resp. u_k] to [0, T] by constancy and continuity (resp., by zero outside [0, T]). The first step is to construct the limit state (this is just the classical argument of the filippov theorem; see Ref. 15). Then, the convergence in the control comes from Assumption (H4). Finally, the control constraint (22) passes to the weak limit by the Banach–Steinhaus theorem in order that the value function is right-continuous.

Hence, $(\rho_k)_k$ being our decreasing sequence, V is also decreasing(Proposition 3.1), so $(V(\rho_k))_k$ is increasing, bounded from above by $V(\rho)$, and thus converging toward $v \le V(\rho)$. Since A is compact, so is

$$N_{\rho_0} = \{(t, x, u) \in \mathbb{R} \times M \times \mathbb{R}^m | (t, x) \in A, u \in U_{\rho_0}(t, x)\};$$

one can find a positive constant K such that

$$|\dot{x}_k| = |f(t, x_k, u_k)| \le K, \qquad t \in [0, t_{f_k}], k \in \mathbb{N},$$

because f is continuous and because the sequence $(N_{\rho_k})_k$ is decreasing. Hence, (x_k) is equilipschitzian when extended to [0, T] by constancy and continuity: up to a subsequence, (x_k) converges uniformly toward x (also Lipschitz) by the Ascoli theorem [the trajectories stay into a fixed compact by virtue of (H2)]. Likewise, we can assume that $(t_{f_k})_k \subset [0, T]$ converges toward some positive t_f. Obviously, $x(0) = x^0$, $(t, x) \in A$ (A is closed), $h(t_f, x(t_f)) = 0$ for $h(t_{f_k}, x_k(t_{f_k})) = 0$, and $(x_k)_k$ is equicontinuous. Finally, as for any k in $\mathbb{N}$,

$$\dot{x}_k \in Q_{\rho_k}(t, x_k) \subset Q_{\rho_0}(t, x_k), \qquad t \in [0, t_{f_k}],$$

we know that $\dot{x} \in Q_{\rho_0}(t, x)$ (closure Theorem 8.6.i of Ref. 15). Then, let u be a measurable selector in $\mathrm{L}_m^\infty([0, t_f])$ (extended to [0, T] by 0 outside [0, t_f]) such that

$$\dot{x} = f(t, x, u), u \in U_{\rho_0}(t, x).$$

By virtue of (H4), there are smooth functions R and S such that, if χ_k denotes the indicator function of $[0, t_{fk}]$,

$$u_k = (R(t, x_k)\dot{x}_k + S(t, x_k))\chi_k, \tag{24}$$

$$u = R(t, x)\dot{x} + S(t, x), \tag{25}$$

on $[0, T]$. Besides, x_k converges to x uniformly, so that $\dot{x}_k$ tends to $\dot{x}$ in the Schwartz distributions sense, $\dot{x}$ being in $L_n^\infty([0, T])$. For all k, since $\dot{x}_k \in Q_{\rho_k}(t, x_k)$, $(\dot{x}_k)_k$ is bounded in $L_n^\infty([0, T])$: as in (23), $(\dot{x}_k)_k$ is equicontinuous and weakly∗converges to $\dot{x}$ in $L_n^\infty([0, T])$. Then, let φ be in $L_m^1([0, T])$; since R and S are continuous, $R(t, x_k)^T\varphi\chi_k \to R(t, x)^T\varphi$ and $(S(t, x_k)\chi_k|\varphi)$ tends to $(S(t, x)|\varphi)$ when $k \to \infty$ in $L_n^1([0, T])$ and $L^1([0, T])$, respectively, by dominated convergence. Thus, $S(t, x_k)\chi_k \to S(t, x)$ weakly* and, $(\dot{x}_k)_k$ being equicontinuous,

$$\int_{[0,T]} (\dot{x}_k | R(t, x_k)^T\varphi\chi_k)\, dt \to \int_{[0,T]} (\dot{x} | R(t, x)^T\varphi)\, dt,$$

that is,

$$\int_{[0,T]} ((R(t, x_k)\dot{x}_k + S(t, x_k))\chi_k | \varphi)\, dt \to$$

$$\int_{[0,T]} (R(t, x)\dot{x} + S(t, x) | \varphi)\, dt.$$

From (24)–(25), we get the weak convergence of u_k toward u. Finally, since $(u_k)_k$ is bounded and weakly∗convergent,

$$\|u\|_\infty \le \liminf_k \|u_k\|_\infty \le \rho, \qquad \text{as } \rho_k \to \rho,$$

by the Banach–Steinhaus theorem. As a consequence, u belongs to $U_{\rho_0}(t, x) \cap B_c(0, \rho) = U_\rho(t, x)$ and (t_f, x, u) is admissible for the limit problem $(\mathrm{OCP})_\rho$. Now, by equicontinuity, we have that $g(t_{fk} \cdot x_k(t_{fk})) \to g(t_f, x(t_f))$, from where we get that

$$(g(t_f), x(t_f)) = v \le V(\rho),$$

since

$$V(\rho_k) = g(t_{f_k}, x_k(t_{f_k})).$$

Necessarily, $v = V(\rho)$ and (t_f, x, u) is solution of $(\mathrm{OCP})_\rho$. Accordingly, $V(\rho_k) \to V(\rho)$, whence we conclude that V is right continuous. □

Besides the regularity of the value function, we obtain a convergence result of the sequence of triples (t_{f_k}, x_k, u_k). In particular, the weak$*$convergence in u implies the classical weak convergence of the controls in $L^1_m([0, T])$ (Ref. 15): for any φ in $L^\infty_m([0, T])$, up to a subsequence,

$$\int_{[0,T]} (u_k|\varphi)\, dt \to \int_{[0,T]} (u|\varphi)\, dt.$$

All these results apply to the orbital transfer problem: one has just to consider the original model where the mass is not explicited as a function of time [Eq. (7), not to introduce $T_{\max}$ into the dynamics as in (14)]. Indeed, we know from Section 2 that, whatever the thrust, the problem is controllable. Moreover, we have also mentioned that the trajectories (mass included) can be proved to stay in a fixed compact: since the dynamics is convex, Assumptions (H1)–(H3) are fulfilled. Now, concerning (H4), since the matrix $B=[f_1, f_2, f_3]$ is an embedding, we can write u as smooth function of $\dot{x}, x, m$ according to

$$u = m\,(B(x)^T B(x))^{-1} B(x)^T [\dot{x} - f_0(x)].$$

As a result, $T_{\max}$ being the parameter ρ, the value function $T_{\max} \mapsto t_f(T_{\max})$ is right continuous and we have the associated weak convergence results.

Remark 3.1. These properties are unchanged if we consider an additional cone constraint on the control (see Ref. 18): instead of taking $U(t, x) = \mathbb{R}^m$ as before, one considers the new constraint (the control has to stay in a cone of angle α and of vertex the axis s of the frame attached to the ellipse; see Fig. 2). If the angle is smaller than $\pi/2$, the control set is still convex and the previous results hold provided we assume controllability.

The next section defines a setting in which it is possible to obtain more regularity of the value function in the transfer case.

4. Differentiability with Respect to the Maximum Thrust

In this section, the idea is to show that, under suitable assumptions, the minimum time for the orbit transfer problem is continuously differentiable with respect to the maximum thrust. To this end, we begin by

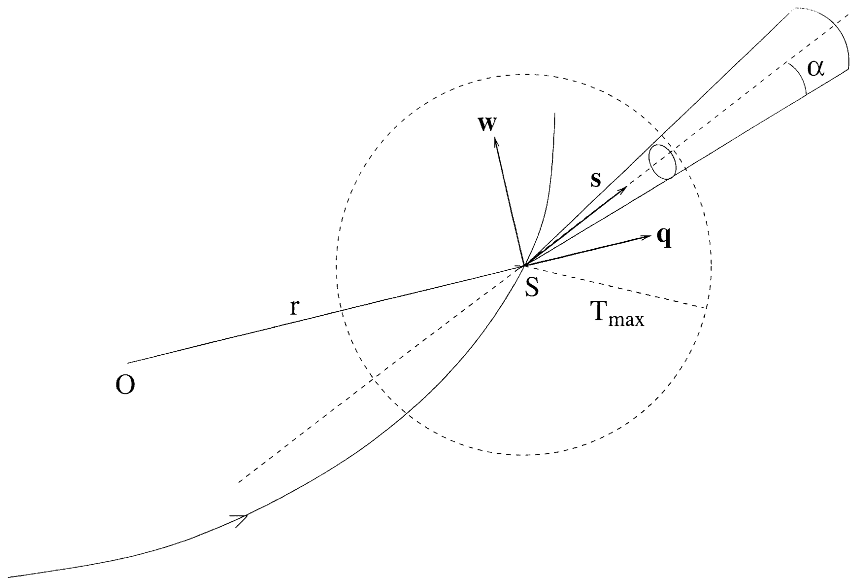

Fig. 2. Cone constraint on the control.

recasting the problem on [0, 1], treating the free final time as an additional constant state variable (this transformation is classical, see e.g. Ref. 19):

$$\min t_f(1), \tag{26a}$$

$$(x, t_f) \in W_7^{1,\infty}([0, 1]), \qquad u \in L_3^{\infty}([0, 1]), \tag{26b}$$

$$\dot{x} = t_f \left\{ f_0(x) + [1/m(t_f t)] \sum_{i=1}^{3} u_i f_i(x) \right\}, \qquad t \in [0, 1], \tag{26c}$$

$$\dot{t}_f = 0, \tag{26d}$$

$$x(0) = x^0, \qquad h(x(1)) = 0, \tag{26e}$$

$$|u| \leq T_{\max}. \tag{26f}$$

We have used hypotheses (A1)–(A3) of Section 2 so as to neglect the path constraints (8–9) and to write the mass as an explicit function of time [Eq. (14)]. We shall still refer to (26) as $(\mathrm{SP})_{T_{\max}}$. Under these assumptions, the Pontryagin maximum principle applies as in Section 2 (but here on [0, 1]) and, if (x, t_f, u) is solution of $(\mathrm{SP})_{T_{\max}}$, there are absolutely continuous multipliers $p = (p_P, p_{e_x}, p_{e_y}, p_{h_x}, p_{h_y}, p_L)$ and p_{t_f} associated to x and t_f, respectively, such that (x, t_f, p, p_{t_f}) is the solution of the two-point boundary-value problem

$$\dot{x} = \partial_p H(t, x, t_f, p, u(y)), \qquad t \in [0, 1], \tag{27a}$$

$$\dot{t}_f = 0, \tag{27b}$$

$$\dot{p} = -\partial_x H(t, x, t_f, p, u(y)), \tag{27c}$$

$$\dot{p}_{t_f} = -\partial_{t_f} H(t, x, t_f, p, u(y)), \tag{27d}$$

with boundary conditions $[p_{t_f}^0 = 0,\ p_L^f = 0,\ p_{t_f}^f = 1$ by transversality and because of (A2)]

$$P(0) = P^0, \qquad P(1) = P^f, \tag{28a}$$

$$e_x(0) = e_x^0, \qquad e_x(1) = e_x^f, \tag{28b}$$

$$e_y(0) = e_y^0, \qquad e_y(1) = e_y^f, \tag{28c}$$

$$h_x(0) = h_x^0, \qquad h_x(1) = h_x^f, \tag{28d}$$

$$h_y(0) = h_y^0, \qquad h_y(1) = h_y^f, \tag{28e}$$

$$L(0) = L^0, \qquad p_L(1) = p_L^f, \tag{28f}$$

$$p_{t_f}(0) = p_{t_f}^0, \qquad p_{t_f}(1) = p_{t_f}^f. \tag{28g}$$

In (27)–(28),

$$H(t, x, t_f, p, u) = t_f \{p \,|\, f_0(x) + [1/m(t_f t)] B(x) u\}$$

is the Hamiltonian of the problem. Of course, the minimization of the Hamiltonian (13) holds unchanged and still defines the control as a smooth function $u(y)$ of $y = (x, p)$ by virtue of (A3) (no-switching assumption) in (27),

$$u(y) = -T_{\max} B(x)^T p / |B(x)^T p|.$$

Let us denote by $\xi(t, x, t_f, p, p_{t_f})$ the right-hand side of (27). The associated maximal flow (Ref. 20) $\phi_t^s(x, t_f, p, p_{t_f})$ is smooth and the boundary-value problem (27)–(28) is equivalent to the so-called shooting equation: find $(p^0, t_f) \in \mathbb{R}^6 \times \mathbb{R}$ such that[11]

$$S(p^0, t_f) = b\,(\phi_1^0(x^0, t_f, p^0, p_{t_f}^0)) = 0. \tag{29}$$

[11]Strictly speaking, we should write t_f^0 instead of t_f in (29), since the latter is treated as a state variable. Nevertheless, as $\dot{t}_f = 0$, we shall not distinguish between the function and the scalar in the sequel.

In (29), the boundary function b is defined by [see (28)]

$$b(x, t_f, p, p_{t_f}) = (P - P^f, e_x - e_x^f, e_y - e_y^f, h_x - h_x^f, h_y - h_y^f, p_L - p_L^f, p_{t_f} - p_{t_f}^f).$$

We know from Section 3 that the minimum time t_f is right-continuous with respect to the maximum thrust $T_{\max}$. Hence, we can use continuation on $T_{\max}$ to get a good initial guess for t_f in (29). Yet, this process is not very efficient numerically: for low thrusts, if $T^c_{\max}$ is the current value of the constraint, the next bound $T^+_{\max}$ has to be taken very close to $T^c_{\max}$ to ensure convergence. Fortunately, we can take advantage of the remarkable heuristic (first reported in Ref. 18) that the minimum time multiplied by the bound on the thrust is nearly constant,

$$t_f \times T_{\max} \simeq \text{const.} \tag{30}$$

Hence, if t_f^c is the minimum time for $T^c_{\max}$, the search for t_f^+ is precisely initialized by $t_f^c T^c_{\max}/T^+_{\max}$. For the other unknown p^0, we have no additional information and we do use mere continuation: if $p^{0,c}$ is the value of the solution adjoint state for $T^c_{\max}$ at $t = 0$, it is used as the initial guess for $p^{0,+}$. Again, this process makes sense only provided the dependence $T_{\max} \mapsto p^0(T_{\max}) = p(0, T_{\max})$ is well defined and has some regularity properties. Moreover, the heuristic (30) clearly suggests a smooth behavior of t_f as a function of $T_{\max}$. Accordingly, it is natural to investigate conditions that would entail, more than continuity, differentiability of the solution with respect to the essential bound on the control as a first step toward a deeper understanding of (30).

To this end, we refer to the work on parametric control of Maurer et al. in Refs. 10, 11, 19. The idea is to extend to the infinite dimension the results of parametric mathematical programming: a family of extremals (that is, points verifying the first-order necessary condition) is constructed, whose (local) optimality is checked by the second-order sufficient conditions. In addition to the usual over-lap between regularity and sufficient conditions, the control setting has its own peculiarities. In particular, for the transfer problem, the no-switching assumption (A3) plays a crucial role: it is at the same time an assumption of regularity of the control, structure for the set of active constraints (with respect to the inequality constraint on the modulus of u), strict complementarity, as well as the strict Legendre–Clebsch condition. Thus, to apply the parametric tools of Ref. 10 to $(\text{SP})_{T_{\max}}$ it is enough to make only two new assumptions of regularity and coercivity (jointly in x and u): for any $T_{\max}$, with $\tilde{x} = (x, t_f)$, $\tilde{p} = (p, p_{t_f})$, $\tilde{y} = (\tilde{x}, \tilde{p})$, and S defined as in (29),

(A4) $[\partial_p S \, \partial_{t_f} S](p(0, T_{\max}), t_f(T_{\max})) \in \mathrm{GL}_7(\mathbb{R})$;

(A5) The following symmetric Riccati equation has a bounded solution:

$$\dot{Q} = -Q\mathscr{A}(t) - \mathscr{A}^T(t)Q + Q\mathscr{B}(t)Q - \mathscr{C}(t), \qquad t \in [0, 1], \tag{31a}$$

$$Q_{77}(0) > 0, \qquad (Q_{ij})_{i,j=6,7}(1) < 0, \tag{31b}$$

with

$$\mathscr{A}(t) = \partial_x \xi_1(t, \tilde{y}(t, T_{\max})),$$

$$\mathscr{B}(t) = \partial_p \xi_1(t, \tilde{y}(t, T_{\max})),$$

$$\mathscr{C}(t) = \partial_x \xi_2(t, \tilde{y}(t, T_{\max})).$$

Since the constraint on the control is a pure one, the Riccati equation in (A5) can be rewritten in a simplified way. For details, we refer to Ref. 19 where the techniques of Ref. 10 are adapted to the free final time problem. In the above form, the equation (31) is well suited for automatic differentiation (see Ref. 21). An example of numerical verification of the coercivity condition (A5) is described in Ref. 4 in a different but still similar case. Here, our aim is rather to provide a formal expression of the derivative $t_f'(T_{\max})$. Indeed, it is a remarkable feature that it is related to the switching function ψ; see Eq. (13).

Proposition 4.1. Under Assumptions (A1)–(A5), the value function t_f is continuously differentiable with respect to $T_{\max}$ and

$$t_f' = -t_f \int_0^1 (d/dt)(t/m)|\psi|\, dt. \tag{32}$$

Proof. Let $T_{\max,0} > 0$ be an arbitrary positive thrust, and let (t_{f_0}, x_0, u_0) be a solution of $(\mathrm{SP})_{T_{\max,0}}$. The associated adjoint states p_0 and $p_{t_f,0}$ are absolutely continuous and, if $\tilde{H}$ is the augmented Hamiltonian,

$$\tilde{H}(t, x, t_f, p, u, \mu) = t_f \left\{ p|f_0(x) + [1/m(t_f t)] \sum_{i=1}^{3} u_i f_i(x) \right\} + (1/2)\mu(|u|^2 - T_{\max}^2),$$

one has

$$\nabla_u \tilde{H}(t, x_0, t_{f_0}, p_0, u_0, \mu_0) = 0,$$

with

$$\mu_0 = t_{f_0}|B(x_0)^T p_0|/(m(t_{f_0}t)T_{\max}, 0). \tag{33}$$

Thanks to Assumption (A3), μ_0 is strictly positive and strict complementarity holds. Moreover, the Hessian $\nabla^2_{uu}\tilde{H} = \mu_0 I_3$ is positive definite (I_3 being the identity matrix of order 3) by the same argument, so that the Legendre–Clebsch condition is also fulfilled. Since the multipliers and the control have obviously the desired smoothness, since we have assumed the regularity and coercivity conditions (A4) and (A5), we apply the results of Ref. 10 and conclude that there exists an open neighborhood of $T_{\max,0}$ on which the sensitivity functions,

$$T_{\max} \mapsto (x, t_f, u)(\cdot, T_{\max}) \in W_7^{1,\infty}([0,1]) \times \mathrm{L}_3^{\infty}([0,1]), \tag{34}$$

$$T_{\max} \mapsto (p, p_{t_f}, \mu)(\cdot, T_{\max}) \in W_7^{1,\infty}([0,1]) \times \mathrm{L}^{\infty}([0,1]), \tag{35}$$

are defined and continuously differentiable. Now, because of (A3), $(\mathrm{SP})_{T_{\max}}$ is equivalent to the abstract parametric optimization problem with equality constraints

$$J(z) \to \min,$$

$$F(z, T_{\max}) = 0,$$

where

$$z = (x, t_f, u) \in Z = W_7^{1,\infty}([0,1]) \times \mathrm{L}_3^{\infty}([0,1]), \qquad J(z) = t_f(1),$$

and

$$F(z, T_{\max}) = \begin{bmatrix} \dot{x} - t_f\{f_0(x) + [1/m(t_f t)] \sum_{i=1}^{3} u_i f_i(x)\} \\ \dot{t}_f \\ x(0) - x^0 \\ h(x(1)) \\ (1/2)(|u|^2 - T_{\max}^2) \end{bmatrix}.$$

The associated Lagrangian (in qualified form) is

$$L(z, \lambda, T_{\max}) = J(z) + \langle \lambda, F(z, T_{\max}) \rangle_{\mathscr{Y}', \mathscr{Y}},$$

with

$$\lambda = (-p, -p_{t_f}, \nu^0, \nu^f, \mu) \in \mathscr{Y}',$$

$$\mathscr{Y} = \mathrm{L}_7^{\infty}([0,1]) \times \mathbb{R}^6 \times \mathbb{R}^5 \times \mathrm{L}^{\infty}([0,1]).$$

By virtue of (34)–(35), the functions $T_{\max} \mapsto z(T_{\max})$ and $T_{\max} \mapsto \lambda(T_{\max})$ (respectively solution and multiplier for the abstract parametric problem) are well defined and continuously differentiable in a neighborhood of

$T_{\max,0}$. Besides, the pair $(z(T_{\max}), \lambda(T_{\max}))$ readily verifies the KKT conditions in this neighborhood,

$$\partial_z L(z(T_{\max}), \lambda(T_{\max}), T_{\max}) = 0,$$

$$\partial_\lambda L(z(T_{\max}), \lambda(T_{\max}), T_{\max}) = 0.$$

Accordingly, the derivative of the value function is

$$t_f'(T_{\max}) = \partial_{T_{\max}} L(z(T_{\max}), \lambda(T_{\max}), T_{\max}).$$

An obvious calculation using the fact that the mass depends explicitly on the parameter $T_{\max}$ [Eq. (14)] yields the desired conclusion. □

Since the mass is decreasing, the derivative t_f' in (32) is clearly negative, in accordance with Proposition 3.1. The last section presents the numerical results obtained with this technique.

5. Numerical Computation

We apply to the minimum time transfer problem the process described in Section 4: the heuristic relation (30) is used to initialize the unknown transfer time, and we do continuation on the initial adjoint state. The first results obtained with this approach on a 2D model are those of Ref. 22. The physical parameters are recalled in Table 1 [more suitable units are chosen, namely Megameters (Mm) and hours (h); compare (11)].

Single shooting is used (see e.g. Ref. 23): the Newton solver is a Hybrid–Powell method, and the ODE solver is a Runge–Kutta integrator of order 4 (Ref. 24). Single shooting is prefered to multiple shooting since a previous study (Ref. 25) in the 2D case showed that multiple shooting, when coupled with continuation, alters the convergence of the process. Indeed, using auxiliary points seems to prevent the current solution from converging toward the solution for the lower thrust (whose structure may be quite different). The right-hand side of the boundary-value problem (27) is computed by automatic differentiation (Ref. 26). The computation,

Table 1. Physical parameters.

P^0	11.625 Mm	P^f	42.165 Mm
e_x^0	0.75	e_x^f	0
e_y^0	0	e_y^f	0
h_x^0	0.612	h_x^f	0
h_y^0	0	h_y^f	0
L^0	π	β	0.0142 Mm^{-1} h
m^0	1500 kg	μ^0	5165.862 $\mathrm{Mm}^3\,\mathrm{h}^{-2}$

Table 2. Minimum transfer times (in hours). Thrusts are in Newtons and computation times (on a 1 Mhz Pentium III) in seconds. The norm of the shooting function S at the solution is also given.

Comput.	T_{max}	t_f	$\|S\|$	Comput	T_{max}	t_f	$\|S\|$
1	60	14.800	1E–8	30	1.4	606.13	1E–10
3	24	34.716	1E–10	43	1	853.31	9E–13
3	12	70.249	9E–9	63	0.7	1214.5	2E–9
7	9	93.272	4E–10	92	0.5	1699.4	1E–8
6	6	141.22	1E–9	155	0.3	2870.2	3E–8
15	3	285.77	4E–10	235	0.2	4265.7	1E–8
22	2	425.61	1E–12	263	0.14	6079.5	7E–9

started for 60 Newtons,[12] is initialized by the results on the coplanar model (Refs. 4–5). The choice of the values of the continuation parameter T_{max} is heuristic. The results are summarized in Table 2. The near constancy of the product $t_f \times T_{max}$ is emphasized by Fig. 3.

The optimal trajectories and controls are given for strong thrusts (60, 12 Newtons) and low thrusts (3 Newtons); see Figs. 4–6. The correction of

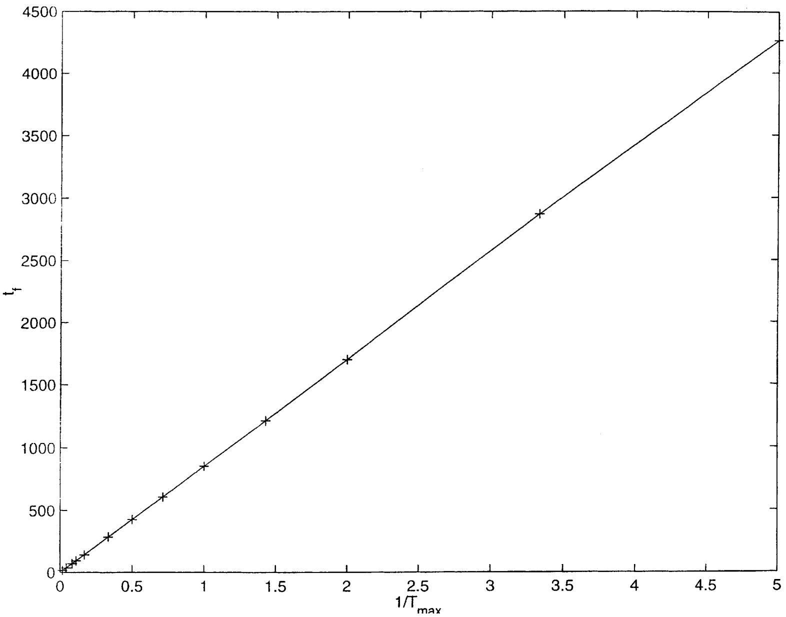

Fig. 3. Near constancy of the product $t_f \times T_{max}$.

[12]Of course, such a thrust is not realistic for a 1500 kilogram satellite: it is used only to ensure convergence of the first homotopy step.

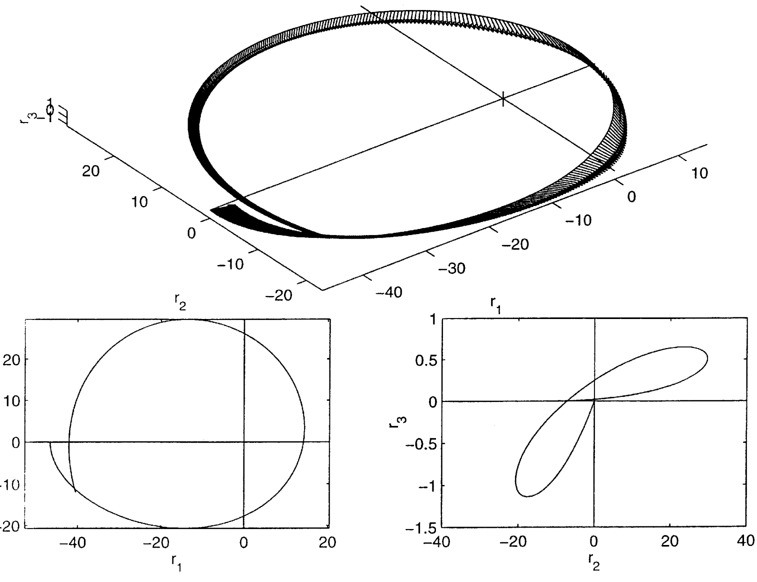

Fig. 4. Optimal trajectories and controls, thrust of 60 Newtons.

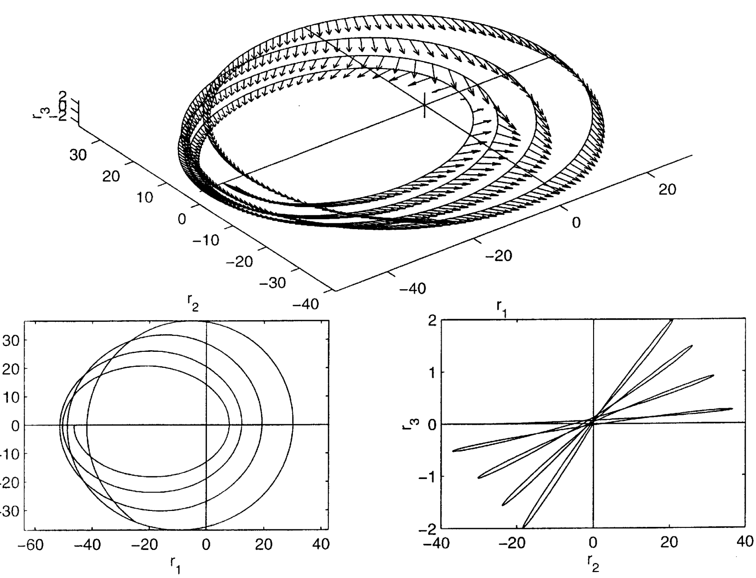

Fig. 5. Optimal trajectories and controls, thrust of 12 Newtons.

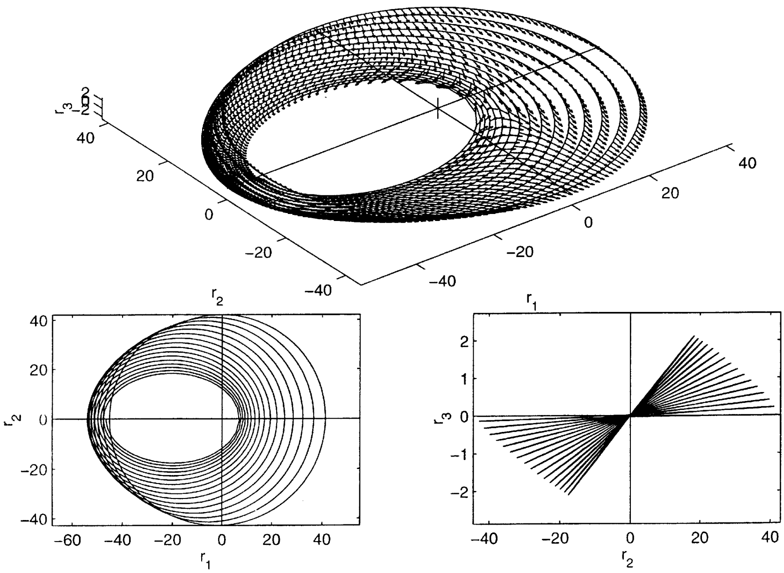

Fig. 6. Optimal trajectories and controls, thrust of 3 Newtons.

the eccentricity is observed in the (r_1, r_2) plane, whereas the change in the inclination is seen in (r_2, r_3). The arrows picture the action of the control. For 0.14 Newton, the result is analogous: the transfer is more than six month long and about 240 revolutions around the Earth are needed. Figure 7 illustrates the properties of the optimal control reported in Section 2: though the control may look discontinuous, there is only one point where μ, the multiplier associated with the control constraint and proportional to the norm of the switching function [see Eq. (33)], is close to zero but not vanishing.

6. Conclusions

We have considered in this paper the minimum time transfer of a satellite from a low orbit toward a high geosynchronous orbit. The transfer is 3D, since both the eccentricity and the inclination have to be corrected. As we have chosen very low thrusts to model an electro-ionic propulsion, the resulting transfers are very long (e.g., more than six months for 0.14 Newton). In order to justify the use of continuation on the thrust, we have studied the dependence of the solution with respect to the essential bound on the control. First, it has been proved that the value function is right

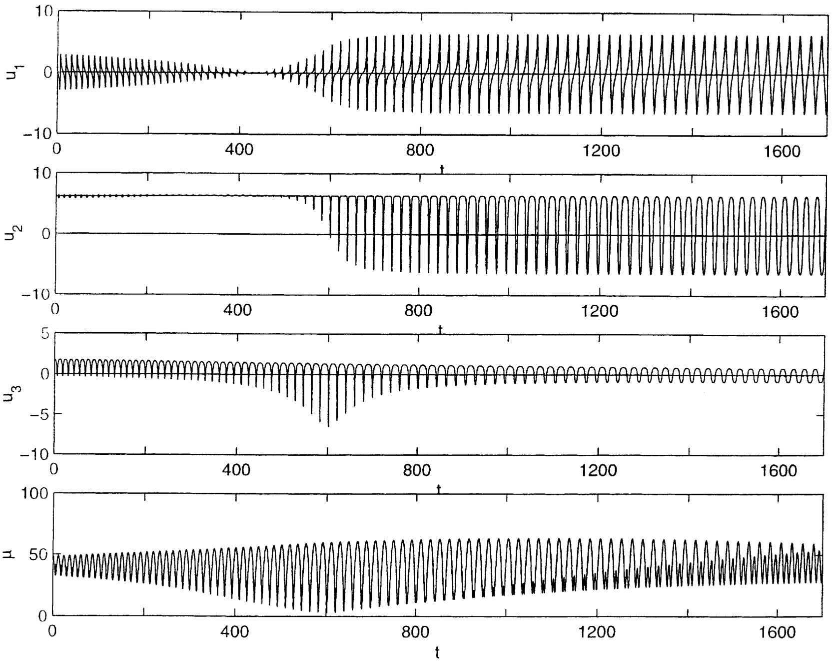

Fig. 7. Structure of the control, thrust of 0.5 Newton.

continuous. Since more is needed in practice, we have also shown to what extent parametric control is relevant here: in particular, we have given a closed expression for the derivative of the value function. This can be viewed as a first attempt to understand the experimental constancy of the product $t_f \times T_{\max}$. Actually, whereas it is probably false that this quantity is constant, we can conjecture that it is asymptotically preserved (asymptotic conservation of the momentum[13]),

$$t_f \times T_{\max} \to c, \qquad T_{\max} \to 0,$$

with c a positive constant depending on only the boundary conditions defined by x^0, m^0, and h [Eq. (10)].

Numerically, we have used single shooting, which proved to be very efficient when combined with (30) and continuation on the adjoint state. Another approach that is not based on (30) is proposed in Ref. 4. New models for the transfer that integrate other constraints (e.g., on the thrust

[13]The product $t_f \times T_{\max}$ is homogeneous to a momentum (product of mass and velocity).

angle) or have different criteria (e.g., maximization of the mass) are under consideration.

References

1. CAILLAU, J. B., GERGAUD, J., and NOAILLES, J., *Continuation Technique for a Weakly Controlled Satellite*, Invited Poster Session, Nonlinear Analysis 2000, Courant Institute, New York, NY, May 28–31, 2000.
2. BAUMANN, H., and OBERLE, H. J., *Numerical Computation of Optimal Trajectories for Coplanar, Aeroassisted Orbital Transfer*, Journal of Optimization Theory and Applications, Vol. 107, pp. 457–479, 2000.
3. NOAILLES, J., and LE, C. T., *Contrôle en Temps Minimal et Transfert Orbital à Faible Poussée*, Équations aux Dérivées Partielles et Applications, Gauthiers-Villars, Paris, France, pp. 705–724, 1998.
4. CAILLAU, J. B., and NOAILLES, J., *Sensitivity Analysis for Time-Optimal Orbit Transfer*, Optimization, Vol. 49, pp. 327–350, 2001.
5. CAILLAU, J. B., and NOAILLES, J., *Coplanar Control of a Satellite around the Earth*, ESAIM Control, Optimisation and Calculus of Variations, Vol. 6, pp. 239–258, 2001.
6. FERRIER, C., and EPENOY, R., *Optimal Control for Engines with Electro-Ionic Propulsion under Constraint of Eclipse*, Acta Astronautica, Vol. 48, pp. 181–192, 2001.
7. GEFFROY, S., EPENOY, R., and NOAILLES, J., *Averaging Techniques in Optimal Control for Orbital Low-Thrust Transfers and Rendezvous Computation*, 11th International Astrodynamics Symposium, Gifu, Japan, pp. 166–171, 1996.
8. BULIRSCH, R., MONTRONE, F., and PESCH, H. J., *Abort Landing in the Presence of Windshear as a Minimax Optimal Control Problem, Part 2: Multiple Shooting and Homotopy*, Journal of Optimization Theory and Applications, Vol. 70, pp. 223–254, 1991.
9. OBERLE, H. J., and TAUBERT, K., *Existence and Multiple Solutions of the Minimum-Fuel Orbit Transfer Problem*, Journal of Optimization Theory and Applications, Vol. 95, pp. 223–262, 1997.
10. MAURER, H., and MALANOWSKI, K., *Sensitivity Analysis for Parametric Optimal Control Problems with Control-State Constraints*, Computational Optimization and Applications, Vol. 5, pp. 253–283, 1996.
11. MALANOWSKI, K., *Sufficient Optimality Conditions for Optimal Control Subject to State Constraints*, SIAM Journal on Control and Optimization, Vol. 35, pp. 205–227, 1997.
12. BETTS, J. T., *Practical Methods for Optimal Control Using Nonlinear Programming*, SIAM, Philadelphia, Pennsylvania, 2001.
13. ZARROUATI, O., *Trajectoires Spatiales*, CNES–Cepadues, Toulouse, France, 1987.

14. CAILLAU, J. B., *Contribution à l'Étude du Contrôle en Temps Minimal des Transferts Orbitaux*, PhD Thesis, Institut National Polytechnique, Toulouse, France, 2000.
15. CESARI, L., *Optimization Theory and Applications*, Springer, New York, NY, 1983.
16. JURDJEVIC, V., *Geometric Control Theory*, Cambridge University Press, Cambridge, England, 1997.
17. SCHWARTZ, L., *Théorie des Distributions*, Hermann, Paris, France, 1973.
18. LE, C. T., *Contrôle Optimal et Transfert Orbital en Temps Minimal*, PhD Thesis, Institut National Polytechnique, Toulouse, France, 1999.
19. MAURER, H., and OBERLE, H. J., *Second-Order Sufficient Conditions for Optimal Control Problems with Free Final Time: The Riccati Approach*, Preprint, 2002.
20. BERGER, M., and GOSTIAUX, B., *Géométrie Différentielle*, Armand-Colin, Paris, France, 1972.
21. CAILLAU, J. B., and NOAILLES, J., *Continuous Optimal Control Sensitivity Analysis with AD*, Automatic Differentiation 2000: From Simulation to Optimization, Edited by G. Corliss, C. Faure, A. Griewank, L. Hascoët, and U. Naumann, Springer Verlag, New York, NY, 2001.
22. MONNERAT, D., *Résolution Numérique des Problèmes d'Optimisation de Trajectories par des Méthodes Homotopiques*, Diploma Thesis, Institut National Polytechnique, Toulouse, France, 1996.
23. OBERLE, H. J., and GRIMM, W., *BNDSCO: A Program for the Numerical Solution of Optimal Control Problems*, Report 515, Institute for Flight Systems Dynamics, German Aerospace Research Establishment (DLR), Oberpfaffenhofen, Germany, 1989.
24. ASCHER, U. M., MATTHEIJ, R. M. M., and RUSSEL, R. D., *Numerical Solution of Boundary-Value Problems for Differential Equations*, Prentice Hall, Englewood Cliffs, New Jersey, 1988.
25. CAILLAU, J. B., GERGAUD, J., and NOAILLES, J., *Trajectoires Optimales à Poussée Continue*, Technical Report, CNES–ENSEEIHT Contract R & T A3006, 1998.
26. BISCHOF, C., CARLE, A., KLADEM, P., and MAUER, A., *Adifor 2.0: Automatic Differentiation of Fortran 77 Programs*, IEEE Computational Science and Engineering, Vol. 3, pp. 18–32, 1996.