# Remarks on Quadratic Hamiltonians in Spaceflight Mechanics

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**Summary.** A particular family of Hamiltonian functions is considered. Such functions are quadratic in the moment variables and arise in spaceflight mechanics when the averaged system of energy minimizing trajectories of the Kepler equation is computed. An important issue of perturbation theory and averaging is to provide integrable approximations of nonlinear systems. It turns out that such integrability properties hold here.

**Key words:** Controlled Kepler equation, averaging, quadratic Hamiltonians, Riemannian problems.

## 1 Introduction

As explained in [4], the energy minimizing trajectories of the controlled Kepler equation [7] can be approximated by trajectories of an *averaged* system. The coplanar single-input system we consider is

$$\ddot{q} = -\mu \frac{q}{r^3} + u \frac{\dot{q}}{|\dot{q}|} \tag{1}$$

for the energy performance index $\int_0^{t_f} |u|^2 \to \min$, where $q$ is the position vector in $\mathbf{R}^2$ and $r$ the radius $(q_1^2 + q_2^2)^{1/2}$. Except in §5 where the real system is considered for the numerical computations, the gravitation constant $\mu$ will be normalized to one in the text. In accordance with (1), the thrust is directed along the speed $\dot{q}$—that is *tangential*—, and if we restrict ourselves to bounded trajectories, the state space in coordinates $(q, \dot{q})$ is the four-dimensional manifold $Q = \{r > 0,\ \dot{q}^2/2 - 1/r < 0\}$. Like the system with two inputs, this single-input model is shown to be controllable and is physically important since it is interpretated as the limit of cone-constrained problems where the control has to remain in a cone directed by the velocity of the spacecraft. In modern applications such as low-thrust orbit transfer [7], controls are very small and act as perturbations not only in (1), but also in the Hamiltonian system provided by Pontryagin maximum principle and describing the extremals with respect to the energy criterion. Such a point of view is standard in celestial mechanics, see for instance [9]. In the framework of Hamiltonian systems, one defines the averaged dynamical system [1] to retrieve the action of the perturbation up to first order.

The averaging process is presented in §2. Then we recall in §3 how a Riemannian control problem can be associated with a Hamiltonian such as the averaged one, which is quadratic in the moment variable $p$. Averaging is intended to provide an integrable approximation of the perturbation of an integrable system, here the coplanar Kepler equations. Integrability of the canonical equations is addressed in §4, first for a two-dimensional subsystem, then for the full system with three degrees of freedom. We end by providing in §5 energy minimizing trajectories obtained by continuation from the averaged system.

## 2 Averaged system

Defining the feedback $u' = u/|v|$ where $v$ stands for $\dot{q}$, the system (1) is written as an affine single-input control one,

$$\dot{x} = F_0(x) + u' F_1(x), \tag{2}$$

where $x$ is the state $(q, \dot{q})$, and where

$$F_0 = -\frac{q}{r^3}\frac{\partial}{\partial v}, \quad F_1 = v\frac{\partial}{\partial v}\cdot$$

Computing Lie brackets of the two vector fields up to length three, one checks that [5]

$$\begin{aligned}
[F_0, F_1] &= -v\frac{\partial}{\partial q} - \frac{q}{r^3}\frac{\partial}{\partial v},\\
[F_1, [F_0, F_1]] &= -F_0,\\
[F_0, [F_0, F_1]] &= \frac{2q}{r^3}\frac{\partial}{\partial q} + \frac{2}{r^5}\left[(2q_1^2 - q_2^2)v_1 + 3q_1q_2v_2\right]\frac{\partial}{\partial v_1}\\
&\quad + \frac{2}{r^5}\left[3q_1q_2v_1 + (2q_2^2 - q_1^2)v_2\right]\frac{\partial}{\partial v_2}\cdot
\end{aligned}$$

Hence, $F_0$, $F_1$, $[F_0, F_1]$ and $[F_0, [F_0, F_1]]$ form a frame, and the system, whose drift $F_0$ is periodic, is controllable [8]. So as to perform averaging, we change coordinates and replace the cartesian ones by an angle, the *longitude* $l$, together with three first integrals of the unperturbed motion: $x = (l, n, e, \omega)$ where $n$ is the *mean movement*, $e$ the *eccentricity*, and $\omega$ the *argument of pericenter* (see, *e.g.*, [9]). The mean movement, the eccentricity and the argument of pericenter define the geometry of the osculating elliptic orbit, while the longitude represents the position on the ellipse. In these coordinates, $Q = \{n > 0,\ e < 1\}$, and

$$\dot{l} = \frac{n[1 + e\cos(l - \omega)]^2}{(1 - e^2)^{3/2}},$$

in order that trajectories can be reparameterized by the cumulated longitude. Before doing so, we consider the system (2) with performance index

$$\int_0^{t_f} u^2 dt = \int_0^{t_f} u'^2 |v|^2 dt \to \min,$$

fixed endpoints, and free final time $t_f$. More precisely, the final cumulated longitude, $l \in \mathbf{R}$, is fixed to $l_f$. For $l_f$ big enough, there are admissible trajectories. Indeed, since the system is controllable, there are trajectories such that $l(t_f) = l_f \bmod 2\pi$. Then, setting the control to zero, $n$, $e$ and $\omega$ remain unchanged while the longitude is increased until it reaches the desired $l_f$.

**Proposition 1.** *The problem of the minimization of energy with fixed cumulated longitude and free final time admits no abnormal extremals.*

*Proof.* The Hamiltonian of the problem is $H = p^0 u'^2 v^2 + H_0 + u' H_1$. For an abnormal extremal, $p^0$ is zero, and $H_1 = 0$ because of the maximization condition of Pontryagin maximum principle. Then, by differentiation, $\{H_0, H_1\} = \{H_0, \{H_0, H_1\}\} + u'\{H_1, \{H_0, H_1\}\} = 0$. Since the final time is free, $H = 0$ so that $H_0$ is also zero. As $[F_1, [F_0, F_1]]$ is colinear to $F_0$, $\{H_1, \{H_0, H_1\}\}$ is also zero. Using the fact that $F_0$, $F_1$, $[F_0, F_1]$ and $[F_0, [F_0, F_1]]$ form a frame, we get the contradiction. □□□

It is shown in [4] that, up to a renormalization, the averaged Hamiltonian associated to normal extremals of the system reparameterized by longitude is

$$H = \frac{1}{2n^{5/3}} \left[ n^2 p_n^2 + \frac{4}{9} \frac{(1 - e^2)^{3/2}}{1 + (1 - e^2)^{1/2}} p_e^2 + \frac{4}{9} \frac{1 - e^2}{1 + (1 - e^2)^{1/2}} \frac{p_\omega^2}{e^2} \right], \quad (3)$$

which defines a quadratic form of full rank with respect to the moment $p$. The underlying Riemannian problem is presented in the next section.

## 3 Associated Riemannian metric

The averaged system can be seen as a rotating deformable solid. The first and second coordinates define the geometry of the solid, an ellipse of given eccentricity $e$ and semi-major axis $a$ ($a^3 n^2 = 1$)—actually, up to a homothety, the geometry is defined by $e$ alone—, while the third one, $\omega$, fixes the angle of rotation around its center. As we are now going to see, the system is associated with the Riemannian problem whose distribution is

$$\dot{n} = u_1 n^{1/6},$$
$$\dot{e} = u_2 \frac{g(e)^{1/2}}{n^{5/6}},$$
$$\dot{\omega} = u_3 \frac{k(e)^{1/2}}{n^{5/6}},$$

so that the cyclicity of $\omega$ implies that there is coupling between the deformation and the rotation: the geometry acts on the rotation, not the converse. The averaged Hamiltonian is clearly normalized to

$$H(q,p) = \frac{1}{2} x^{\alpha} \left( x^2 p_x^2 + g(y) p_y^2 + k(y) p_z^2 \right) \tag{4}$$

setting $\alpha = -5/3$, $g(y) = (4/9)(1-y^2)^{3/2}/[1+(1-y^2)^{1/2}]$, and $k(y) = [4/(9e^2)](1-y^2)/[1+(1-y^2)^{1/2}]$.

*Remark 1.* In the case of two controls, when the direction of the thrust is not prescribed anymore, the averaged Hamiltonian obtained in [3] also belongs to the same class, still with $\alpha = -5/3$.

For $\alpha$ not zero, the Hamiltonian (4) defines a quadratic form in $p$ parameterized by $q = (x, y, z)$ in $Q = \{x > 0\}$, and this form can be written as a sum of squares [3]. The task is straightforward here since we have *orthogonal coordinates* [2], in order that

$$H = \frac{1}{2} \sum_{i=1}^{3} P_i(q,p)^2$$

with $P_i = \langle p, F_i(q) \rangle$, $i = 1, \ldots, 3$, $F_1 = x^{1+\alpha/2} \partial/\partial x$, $F_2 = x^{\alpha/2} g^{1/2}(y) \partial/\partial y$, and $F_3 = x^{\alpha/2} k^{1/2}(y) \partial/\partial z$. Hence, $H$ can be seen as the Hamiltonian associated with the Riemannian problem [6] with dynamics $\dot{q} = \sum_{i=1}^{3} u_i F_i(q)$, $u = (u_1, u_2, u_3)$ in $\mathbf{R}^3$, and criterion $\int_0^{t_f} |u|^2 dt \to \min$ with prescribed final time (again denoted $t_f$ for the sake of simplicity). Writing the dynamics $\dot{q} = B(q)u$, one has $|u|^2 = \left( (BB^T)^{-1}(q)\dot{q} | \dot{q} \right)$ and the Riemannian metric is

$$ds^2 = \frac{1}{x^{\alpha}} \left( \frac{dx^2}{x^2} + \frac{dy^2}{g(y)} + \frac{dz^2}{k(y)} \right). \tag{5}$$

As $z$ is a cyclic coordinate of the Hamiltonian, the system can be restricted to the two-dimensional subspace $Q_0 = Q \cap \{z = 0\}$. We start by studying integrability on this subspace.

## 4 Integrability

We first compute a normal form of the metric.

**Lemma 1.** *A normal form for the metric (5) of the full system is*

$$ds^2 = du^2 + u^2 \left( dv^2 + \frac{dw^2}{l(v)} \right).$$

*Proof.* Consider the change of coordinates defined by $u = -2/(\alpha x^{\alpha/2})$, $v = \varphi(y)$, $w = z$, where $\varphi$ is the quadrature

$$\varphi = \frac{|\alpha|}{2} \int \frac{dy}{g^{1/2}(y)}.$$

Letting $l = (4/\alpha^2) k \circ \varphi^{-1}$, one gets the desired expression. □□□

**Corollary 1.** *A normal form of the Hamiltonian is*

$$H = \frac{1}{2} p_u^2 + \frac{1}{u^2} H_2$$

*with* $H_2 = (1/2)(p_v^2 + l(v) p_w^2)$.

Restricted to $Q_0 = Q \cap \{w = 0\}$ and $\{p_w = 0\}$, the Hamiltonian becomes $H = (1/2)(p_u^2 + \frac{1}{u^2} p_v^2)$ and the metric is in *polar form*, $ds^2 = du^2 + u^2 dv^2$. It is clearly isometric to the flat metric in dimension two so that, in coordinates $u \cos v$, $u \sin v$, the proposition hereafter holds.

**Proposition 2.** *The geodesics of the two-dimensional subsystem are straight lines.*

Using the complex notation $c = c_1 t + c_2$ to parameterize such lines, we are able to write the geodesics in the original coordinates, $x = [4/(\alpha^2 |c|^2)]^{1/\alpha}$, $y = \varphi^{-1}(\arg(c))$. We address now integrability of the system in dimension three.
Our first step is to reduce the analysis to the study of the *Liouville metric* [2] $ds^2 = dv^2 + dw^2/l(v)$. To this end, we have the following lemma.

**Lemma 2.** *The squared coordinate $u^2$ is a polynomial of degree two in $t$.*

*Proof.* The canonical equations in $(u, p_u)$ are $\dot{u} = \partial H/\partial p_u = p_u$ and $\dot{p}_u = -\partial H/\partial u = 2H_2/u^3$. As a result, $d(u p_u)/dt = p_u^2 + 2H_2/u^2$ that is equal to twice the Hamiltonian. Then, $d^2 u^2/dt^2 = 4H$, whence the conclusion. □□□

The main result follows.

**Proposition 3.** *The full three-dimensional system is integrable by quadratures.*

*Proof.* The two variables $u, p_u$ are computed thanks to the previous lemma. If we reparameterize the system in the remaining terms according to the time change $d\tau = dt/u^2$, we obtain the canonical equations of the auxiliary Hamiltonian $H_2 = (1/2)(p_v^2 + l(v) p_w^2)$ with two degrees of freedom. Since $w$ is a cyclic coordinate, $p_w$ is a first integral in involution with $H_2$ which is then integrable by Liouville theorem. □□□

*Remark 2.* The metric associated with $H_2$, $ds^2 = dv^2 + dw^2/l(v)$, is a Liouville metric, that is a metric of the form $(f(x)+g(y))(dx^2+dy^2)$ since

$$ds^2 = l^{-1}(v)\left[(l^{1/2}dv)^2 + dw^2\right].$$

As such, it is known to be integrable [2]. Moreover, the relevant quadrature is obviously deduced from the canonical equations:

$$\dot{v}^2 + l(v)p_w^2 = \text{constant}.$$

*Remark 3.* Liouville integrability is also obtained by noting that $H$, $H_2$ and $p_w$ are three independent first integrals in involution.

In summary, the three quadratures used to integrate the whole system are:

$$\varphi = \frac{|\alpha|}{2}\int \frac{dy}{g^{1/2}(y)},$$
$$\tau = \int \frac{dt}{u^2(t)},$$
$$\psi = \int \frac{dv}{(1-bl(v))^{1/2}},$$

where $u^2$ is a degree two polynomial in $t$, $l = k \circ \varphi^{-1}$ and $b$ a constant. We end the paper by expliciting some of this computations for the application considered.

## 5 Numerical results

According to the previous sections, the whole system is integrable and the two-dimensional subsystem‡ is flat. The first quadrature can be explicitly computed,

$$\varphi(e) = \frac{5}{4}\arcsin[1-2(1-e^2)^{1/2}], \tag{6}$$

as well as the associated coordinates on $Q_0$.
Namely, in dimension two,

$$n = (25/36)^{3/5}|c|^{6/5}, \tag{7}$$

$$e = \pm\left[1 - \frac{(1-\sin((4/5)\arg c))^2}{4}\right]^{1/2}, \tag{8}$$

‡ This subsystem is important in practice since it corresponds to transfers towards circular orbits.

where $c = c_1 t + c_2$ is, as in §4, a complex polynomial of degree one in $t$. Equation (8) is multiform because the quadrature (6) defines a diffeomorphism either of $]-1,0[$ or $]0,1[$ to $]-5\pi/8, 5\pi/8[$. Even in the two-dimensional case, contact with the boundary of $Q_0$ may occur, either with the parabolic boundary $\{e = 1\}$, or with $\{n = 0\}$ (see [3] for a discussion in the two-input situation). We do not touch this point here and restrict ourselves to complete geodesics. Clearly then, when $t \to \infty$, $n \to n_\infty = \infty$ (that is $a \to 0$, $a$ semi-major axis) and $e \to e_\infty = \pm[1-(1/4)\,(1-\sin((4/5)\arg_\infty))^2]^{1/2}$, where $\arg_\infty = \arg c_1$ when $c_1$ is not zero, $\arg c_2$ otherwise (stationary case). This asymptotic behaviour is summarized in the last proposition.

**Proposition 4.** *The (non-stationary) complete trajectories of the two-dimensional system converge to a collision with a limit value of the eccentricity.*

We end the section with numerical results. Using the boundary conditions of table 1, the analytical solution of the two-dimensional averaged subsystem is used to initialize the computation of energy minimizing trajectories of Kepler equation (1) by a standard shooting method. The gravitation constant is the Earth constant, $\mu = 5165.8620912\ \mathrm{Mm}^3{\cdot}\mathrm{h}^{-2}$, and the target is the geostationary orbit (the initial orbit around the Earth is taken low and eccentric). Results are given for different values of the fixed final time at figures 1 to 3. As the transfer time increases, there are more and more revolutions that osculate intermediate orbits of the averaged motion with a growing accuracy, thus illustrating convergence towards the averaged problem.

**Table 1.** Boundary conditions.

| **Variable** | **Initial cond.** | **Final cond.** |
|---|---|---|
| $n$ | $5.2475e-1\ \mathrm{h}^{-1}$ | $2.6251e-1\ \mathrm{h}^{-1}$ |
| $e$ | 0.75 | 0 |
| $\omega$ | 0 rad | 0 rad |

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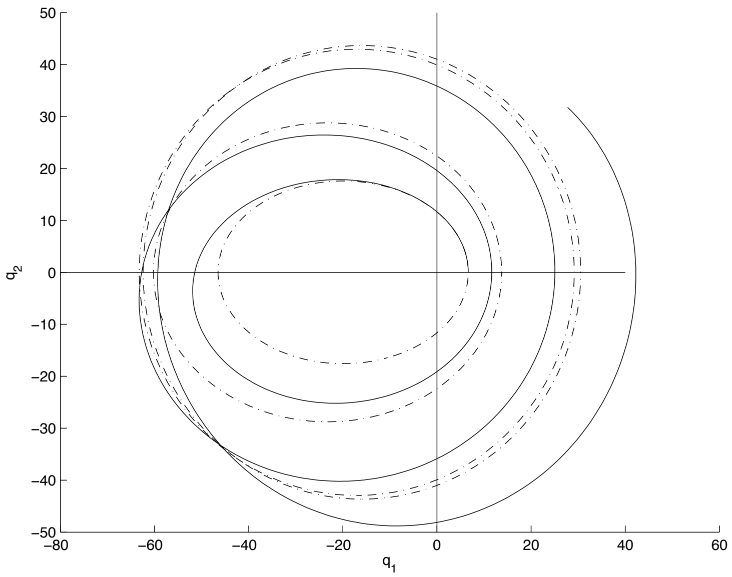


**Fig. 1.** Energy minimizing transfer towards the geostationary orbit, final time $t_f = 19.290$ hours. The trajectory is the solid line that osculates the dashed intermediary orbits of the averaged system.

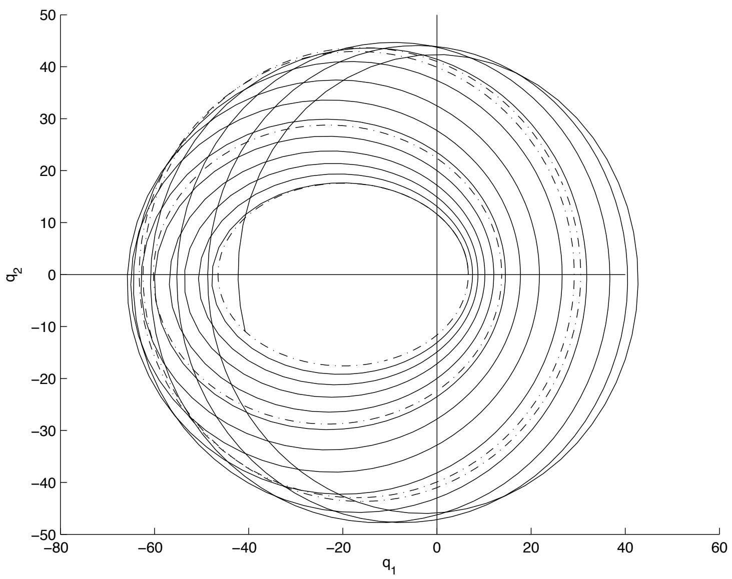


**Fig. 2.** Energy minimizing transfer towards the geostationary orbit, final time $t_f = 77.160$ hours (solid line).

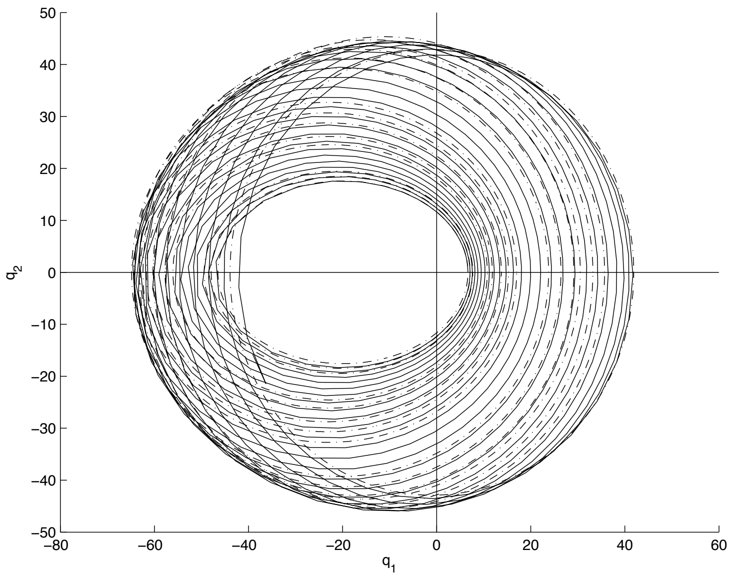


**Fig. 3.** Energy minimizing transfer towards the geostationary orbit, final time $t_f =$ 154.32 hours (solid line).

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