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Zermelo-Markov-Dubins with two trailers

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Abstract: We study the minimum time problem for a simplified model of a ship towing a long spread of cables. Constraints are on the curvature of the trajectory as well as on the shape of what represent the spread of cables here. This model turns out to be the same as a cart towing two trailers and rolling without sleeping on a plane in uniform translation. We analyse the Hamiltonian system describing the extremal flow given by Pontrjagin maximum principle. We detail the equilibria of the system and prove that, contrary to the case of one trailer studied previously by part of the authors, it is not solvable by quadratures. Preliminary numerical results are given.

Keywords: Zermelo navigation, minimum time, turnpike, integrability, Kovacic algorithm

1. INTRODUCTION

The present note takes place in a line of work motivated by the optimization of turns and maneuvers of marine vessels towing a set of long and fragile underwater cables. It is a follow-up to Caillau et al. (2019), where the interested reader can find many more details about motivations in terms of marine seismic acquisition.

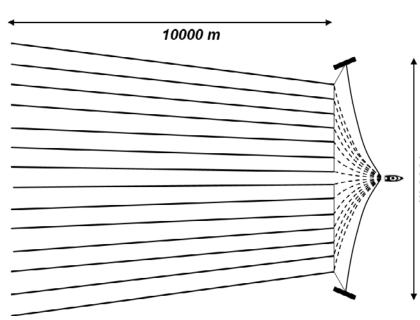

Fig. 1. dimensions of a seismic acquisition spread. Illustration from Caillau et al. (2019).

Each of these ships collects data from the ocean ground via sonic sources and sensors located in a spread of cables it is towing. This can be done only while the ship is sailing along a straight line. In a typical campaign, the ship runs on parallel pre-defined straight lines, and must perform a u-turn at the end of each straight line to position itself at the starting point of the next one, acquisition being stopped during this maneuver. This maneuver is not constrained to follow a specific path, and hence can be optimized. The objective is to perform it in minimum time with given starting and end points, while being gentle enough to preserve the integrity of the spread of streamers. Since a typical u-turn can take more than an hour, minimizing time is important. Integrity of the spread of streamers, a few kilometers long, is primarily a matter of bounding the curvature of the trajectory. The starting and end points are at the end of a straight line and the beginning of the next one respectively; however if the state of the model takes into account the shape of the towed spreader, it should also be specified that it has to be in the right position at the end of the u-turn, i.e. in the relative equilibrium that is asymptotically attained during a straight line. There are also motivations in terms of traffic near airports of unmanned aerial vehicles Techy and Woolsey (2009).

After describing the model in Section 2, we apply Pontrjagin maximum principle: the minimum time extremals of the problem are solutions of a Hamiltonian system and can be regular or singular, as detailed in Section 3. Then, equilibria of the singular extremals are studied in Section 4; among these, hyperbolic points play an important role and are related to the turnpike phenomenon. In Section 5, we give obstructions to solvability by quadratures (integrability) both for the singular and regular extremals thanks to differential Galois theory. Preliminary numerical simulations of the system with two trailers are provided in the final section.

2. MODEL

If one takes into account only the position and orientation of the ship and the constraint is a bound on the curvature of the trajectory, one gets the so called Dubin's problem Dubins (1957): the magnitude of the speed being

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fixed, one seeks the shortest path from a point to another (including direction of the tangent) with a bound on curvature. The maximum curvature has to be small enough in order to preserve the integrity of the towed equipment during the turn. See Dubins (1957); Sussmann and Tang (1991); Boissonnat et al. (1994), and textbooks for Dubin's problem. There are two drawbacks to this approach: it does not take into account possible sea currents, and it does not contain any description of the hydrodynamic behavior of the towed cables (the dynamic equations only contain a kinematic of the ship itself).

Adding the sea current into the problem, still without modelling the cables behavior, leads to a so-called Zermelo-Markov-Dubins problem (the term was apparently coined in Bakolas and Tsiotras (2013)) where it is well documented, see also Techy and Woolsey (2009). In Caillau et al. (2019), we introduced a possible model for the towed cables, consisting in replacing them with a finite number of rigid links or "trailers", their dynamic (in fact kinematic) equations coming either from a simple punctual drag force applied by the ocean to the spread at each "joint", or from mimicking the equations of rolling without slipping, in the frame that moves with the fluid, as if each link was a trailer on wheels (see Jean (1996)). That are two types of models, that we call for short *rolling without slipping* models and *drag* models. Models also differ by the number of links, or number of trailers. For a single trailer, drag coincides with rolling without slipping. The state variables may be chosen as follows: two cartesian coordinates (x, y) and an angle θ for the position and orientation of the ship, or towing vehicle in the vehicle with trailers point of view, plus as many angles as trailers, α_i being the angles between the $(i-1)^{\text{th}}$ and the i^{th} trailer (where the towing vehicle is counted as the 0^{th} trailer).

The case of a single trailer (where, as we just mentioned, drag or rolling without slipping models coincide) was examined in Caillau et al. (2019). There, we explain among other things that this optimal control problem is Liouville integrable. Here we investigate only the case of *two* trailers and "rolling without slipping"; the conclusions do not differ for the "drag" model but we do not present them due to space limitation. They enjoy interesting properties but we prove that they are not integrable, prohibiting an almost explicit resolution.

Since the model is a heuristically approached model for a ship towing a spread of streamers but an *exact* kinematic model for a cart towing two trailers all rolling without slipping for instance on a conveyor belt in translation, and despite the motivation for navigation, we now consider the latter, illustrated in Figure 2, rather than the ship.

Parameters. There are five scalar parameters to this minimum time problem: $W \in \mathbb{R}^2$, the current speed with respect to a frame fixed to the ocean's floor, supposed constant (depends neither on time nor on the point), the magnitude $V_0 > 0$ of the longitudinal speed of the towing vehicle with respect to the mobile frame, the minimum curvature radius $R_{\min} > 0$, and the length $L > 0$ of each link. Via a *rescaling* of time and space, and a rotation that brings W to the $0x$ semi-axis, the magnitude of the longitudinal velocity as well as the maximum curvature (or minimum curvature radius) of the trajectory in the mobile frame become equal to 1, and there remains only two scaled dimension-less parameters:

$$\ell = L/R_{\min}\,, \quad w = \|W\|/V_0\,. \tag{1}$$

We will see in (8) that w and ℓ cannot be too large.

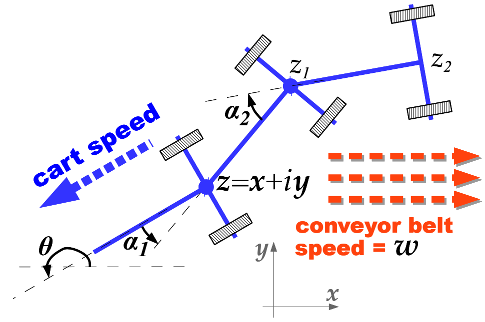

Fig. 2. **A cart with two trailers rolling without slipping on a conveyor belt.** Using complex notations in the plane, $z = x + i\,y$ is the middle of the axle of the cart, whose speed *with respect to the conveyor belt* is $e^{i\theta}$ (magnitude normalized to 1). The control u is the angular velocity $\dot\theta$ of the cart, after re-mormalization. The first trailer is attached to the cart at point z, and its angle with respect to the axis of the cart is α_1, so that the other end of the cart is at point $z_1 = z + \ell e^{i\theta+\alpha_1}$, rolling without sleeping means that the velocity of point z_1 with respect to the conveyor belt is along the axis of the trailer, i.e. has polar angle $\theta + \alpha_1$. The second trailer is attached to the first one at z_1, its angle with respect to first one is α_2, the other end of that second cart is at $z_2 = z_1 + \ell e^{i(\theta+\alpha_1+\alpha_2)}$, and the velocity of z_2 with respect to the conveyor belt has to have polar angle $\theta + \alpha_1 + \alpha_2$.

Equations. Variables $x, y, \theta, \alpha_1, \alpha_2, u$ being defined in Figure 2, the state is $q = (x, y, \theta, \alpha_1, \alpha_2) \in \mathbb{R}^2 \times \mathbb{S}^1 \times (\mathbb{S}^1)^2$ (α_1, α_2 are further restricted below), and the control is $u \in [-1, 1]$. The equations read

$$\begin{cases} \dot x = \cos\theta + w\,, \\ \dot y = \sin\theta\,, \\ \dot\theta = u\,, \\ \dot\alpha_1 = -\,u - \dfrac{\sin\alpha_1}{\ell}\,, \\ \dot\alpha_2 = \dfrac{\sin\alpha_1}{\ell} - \dfrac{\cos\alpha_1\sin\alpha_2}{\ell}\,, \end{cases} \qquad |u| \le 1\,, \tag{2}$$

or

$$\dot q = F_0(q) + uF_1(q)\,, \quad |u| \le 1\,,$$

with

$$F_0 = (\cos\theta + w)\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y} - \frac{\sin\alpha_1}{\ell}\frac{\partial}{\partial\alpha_1} + \left(\frac{\sin\alpha_1}{\ell} - \frac{\cos\alpha_1\sin\alpha_2}{\ell}\right)\frac{\partial}{\partial\alpha_2} \tag{3}$$

and

$$F_1 = \frac{\partial}{\partial\theta} - \frac{\partial}{\partial\alpha_1}\cdot \tag{4}$$

There are constraints on some parameters ans states. First, it is clear that $\dot x$ is non negative if $w \ge 1$, forbidding any kind of controllability. Hence we assume

$$0 \le w < 1\,. \tag{5}$$

Another point is that we want a model where, for any control in the prescribed bound ($|u| \le 1$), the angles α_1, α_2

remain "small" if their initial condition is "small". Let us set $u = \pm 1$ and examine a solution $\alpha_1(t), \alpha_2(t)$ with initial condition close to zero. If $\ell > 1$, α_1 goes unbounded, but if $\ell < 1$ it converges exponentially to $\pm\alpha_1^*$, with

$$\alpha_1^* = \arcsin \ell. \tag{6}$$

We set $\ell < 1$. Fixing $\alpha_1 = \pm\alpha_1^*$, α_2 goes unbounded if $\tan\alpha_1^* > 1$, but converges exponentially, if $\tan\alpha_1^* < 1$, to $\pm\alpha_2^*$ with

$$\alpha_2^* = \arcsin(\tan\alpha_1^*) = \arcsin(\ell/\sqrt{1-\ell^2}). \tag{7}$$

Hence we assume that $\tan\alpha_1^* < 1$, i.e.

$$0 < \ell < \sqrt{2}/2\,. \tag{8}$$

Then, $\mathbb{R}^2 \times \mathbb{S}^1 \times (-\alpha_1^*, \alpha_1^*) \times (-\alpha_2^*, \alpha_2^*)$ is an invariant subset of $\mathbb{R}^2 \times \mathbb{S}^1 \times \left(\mathbb{S}^1\right)^2$ for any control; we chose this restricted state space and assume from now on that

$$q = (x, y, \theta, \alpha_1, \alpha_2) \in \mathbb{R}^2 \times \mathbb{S}^1 \times (-\alpha_1^*, \alpha_1^*) \times (-\alpha_2^*, \alpha_2^*)\,. \tag{9}$$

3. EXTREMALS OF MINIMUM TIME PROBLEM

In order to apply Pontrjagin maximum principle (see *e.g.* the textbook Agrachev and Sachkov (2004)), we define the Hamiltonian of the problem:

$$\begin{aligned} H = p^0 + p_x(\cos\theta + w) + p_y \sin\theta + (p_\theta - p_{\alpha_1})u \\ - \frac{p_{\alpha_1}}{\ell}\sin\alpha_1 + \frac{p_{\alpha_2}}{\ell}(\sin\alpha_1 - \cos\alpha_1 \sin\alpha_2). \end{aligned} \tag{10}$$

Any minimum time trajectory $t \mapsto (q(t), u(t))$ must be the projection of a curve $t \mapsto (q(t), p(t), u(t))$ in the cotangent bundle of the state space, solution of the Hamiltonian system associated with H, with p^0 a non positive constant, a control $u(t)$ that maximizes $u \mapsto H(q(t), p(t), u)$ for almost all time t; H must be identically zero along the solution. In particular, the adjoint state $p = (p_x, p_y, p_\theta, p_{\alpha_1}, p_{\alpha_2})$ is a solution of:

$$\begin{cases} \dot p_x = 0 \\ \dot p_y = 0 \\ \dot p_\theta = p_x \sin\theta - p_y\cos\theta \\ \dot p_{\alpha_1} = \dfrac{p_{\alpha_1}}{\ell}\cos\alpha_1 - \dfrac{p_{\alpha_2}}{\ell}(\cos\alpha_1 + \sin\alpha_1\sin\alpha_2) \\ \dot p_{\alpha_2} = \dfrac{p_{\alpha_2}}{\ell}\cos\alpha_1\cos\alpha_2 \end{cases}$$

The set $\{p_{\alpha_2} = 0\}$ is invariant; the system restricted to this set is equivalent to the case of one trailer only, treated in Caillau et al. (2019); p_{α_2} never vanishes if it is not identically zero. Similarly, $(p_{\alpha_1}, p_{\alpha_2})$ either does not vanish or is identically zero, and the latter case reflects the original problem with zero trailer (see introduction).

As is customary, we use the notations $H_i = \langle p, F_i\rangle$, $i = 0, 1$ (Hamiltonian lifts), as well as $H_{01} = \{H_0, H_1\}$, $H_{001} = \{H_0, H_{01}\}$, and so on... where $\{.,.\}$ stands for the Poisson bracket. Then $H = H_0 + u\,H_1$. Also, in order to replace p^0, p_x, p_y, let us define γ, ρ, ϕ by

$$\gamma = -p^0 - p_x\, w\,, \quad \begin{pmatrix} p_x \\ p_y \end{pmatrix} = \rho \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix}, \ \rho \geq 0. \tag{11}$$

These are constant on any extremal. Pontrjagin's maximum principle also implies a nonzero adjoint vector:

$$(\gamma, \rho, p_\theta, p_{\alpha_1}, p_{\alpha_2}) \neq (0,0,0,0,0). \tag{12}$$

The following identities can be checked via elementary (tedious) computations:

$$H_0 = -\gamma + H_{101} \tag{13}$$

$$H_1 = p_\theta - p_{\alpha_1} \tag{14}$$

$$\begin{aligned} H_{01} = -\frac{p_{\alpha_1}}{\ell}\cos\alpha_1 + \frac{p_{\alpha_2}}{\ell}(\cos\alpha_1 + \sin\alpha_1\sin\alpha_2) \\ - \rho\sin(\phi - \theta) \end{aligned} \tag{15}$$

$$\begin{aligned} H_{101} = -\frac{p_{\alpha_1}}{\ell}\sin\alpha_1 + \frac{p_{\alpha_2}}{\ell}(\sin\alpha_1 - \cos\alpha_1\sin\alpha_2) \\ + \rho\cos(\phi - \theta) \end{aligned}$$

$$H_{001} = \frac{-p_{\alpha_1} + p_{\alpha_2}(1 + \cos\alpha_2)}{\ell^2} \tag{16}$$

Solutions are concatenations of *regular arcs*, or *bang arcs*, on which H_1 is nonzero, hence $u = \text{sign} H_1$, for almost all time, and of *singular arcs*, on which H_1 is identically zero, and that we particularly study in the rest of this section.

Since $\dot H_1 = H_{01}$ and $\dot H_{01} = H_{001} + uH_{101}$, one has

$$H_1 = H_0 = H_{01} = H_{001} + u\,H_{101} = 0$$

identically on a singular arc ($H_0 = 0$ because $H = 0$ on any minimum time solution). The three first equality restrict the flow to some co-dimension 3 subset and the last one yields the control if $H_{101} \neq 0$.

Let us prove that this is always the case, i.e. that *all singular arcs are of order one* (see Agrachev and Sachkov (2004)). Indeed, if H_{101} (equal to γ according to (13)) is zero, so must be H_{001}. A computation yields

$$\dot H_{001} = H_{0001} = \frac{\cos\alpha_1}{\ell^3}\left(-p_{\alpha_1} + (2 + \cos\alpha_2)p_{\alpha_2}\right),$$

that must also be zero; this and (16) implies $p_{\alpha_2}\cos\alpha_1 = 0$. Since $\cos\alpha_1$ does not vanish in the state space, see (9), (6) and (8), p_{α_2} must be identically zero, which yields a contradiction to (12).

Hence $\gamma \neq 0$ and the control is given by $u = -H_{001}/\gamma$. The description of the extremals reduces to

$$\begin{cases} \dot\alpha_1 = -\dfrac{\sin\alpha_1}{\ell} - \dfrac{p_{\alpha_1}}{\gamma\ell^2} + \dfrac{p_{\alpha_2}}{\gamma\ell^2}(1 + \cos\alpha_2), \\ \dot\alpha_2 = \dfrac{1}{\ell}(\sin\alpha_1 - \cos\alpha_1\sin\alpha_2), \\ \dot p_{\alpha_1} = \dfrac{p_{\alpha_1}}{\ell}\cos\alpha_1 - \dfrac{p_{\alpha_2}}{\ell}(\cos\alpha_1 + \sin\alpha_1\sin\alpha_2), \\ \dot p_{\alpha_2} = \dfrac{p_{\alpha_2}}{\ell}\cos\alpha_1\cos\alpha_2. \end{cases} \tag{17}$$

The six other variables are recovered from $\alpha_1, \alpha_2, p_{\alpha_1}, p_{\alpha_2}$ either through $H_0 = H_1 = H_{01} = 0$ or by quadrature.

4. ANALYSIS OF EQUILIBRIA OF SINGULAR FLOW

Lemma 1. System (17) has 8 equilibria (with angles taken modulo 2π) given by the 4-tuple

$$A = \{(0,0,0,0), (\pi,0,0,0), (0,\pi,0,0), (\pi,\pi,0,0)\}$$

and the 4-tuple

$$B = \left\{ \left(\frac{\pi}{4}, \frac{\pi}{2}, -\sqrt{2}\ell\gamma, -\frac{\ell\gamma}{\sqrt{2}}\right), \left(\frac{3\pi}{4}, -\frac{\pi}{2}, -\sqrt{2}\ell\gamma, -\frac{\ell\gamma}{\sqrt{2}}\right), \left(-\frac{\pi}{4}, \frac{\pi}{2}, \sqrt{2}\ell\gamma, \frac{\ell\gamma}{\sqrt{2}}\right), \left(-\frac{3\pi}{4}, -\frac{\pi}{2}, \sqrt{2}\ell\gamma, \frac{\ell\gamma}{\sqrt{2}}\right) \right\}.$$

Proof. Equilibria of the system are deduced by direct analysis as follows.

- From $\dot p_{\alpha_2} = 0$, either $p_{\alpha_2} = 0$ or $\cos\alpha_2 = 0$. Having $\cos\alpha_1 = 0$ is forbidden by $\dot\alpha_2 = 0$, however.

- If $p_{\alpha_2} = 0$, then $p_{\alpha_1} = 0$ follows from $\dot{p}_{\alpha_1} = 0$. $\dot{\alpha}_1 = 0$ and $\dot{\alpha}_2 = 0$ then yield $\sin\alpha_1 = \sin\alpha_2 = 0$. This implies equilibria in family A
- On the other hand, if $\cos\alpha_2 = 0$, then $\alpha_2 = \pm\pi/2 + 2k\pi$, $k \in \mathbb{Z}$. We now show these equilibria correspond to family B.
- If $\alpha_2 = \pi/2 + 2k\pi$, $k \in \mathbb{Z}$, then $\dot{\alpha}_2 = 0$ implies that $\cos\alpha_1 = \sin\alpha_1$, and $\dot{p}_{\alpha_1} = 0$ implies that $p_{\alpha_1} = 2p_{\alpha_2}$. Finally, $\dot{\alpha}_1 = 0$ implies that $p_{\alpha_2} = -\ell\gamma\cos\alpha_1 = \pm\ell\gamma/\sqrt{2}$.
- If $\alpha_2 = -\pi/2 + 2k\pi$, $k \in \mathbb{Z}$, then $\dot{\alpha}_2 = 0$ now implies that $\cos\alpha_1 = -\sin\alpha_1$, and $\dot{p}_{\alpha_1} = 0$ implies again that $p_{\alpha_1} = 2p_{\alpha_2}$. Finally, $\dot{\alpha}_1 = 0$ implies that $p_{\alpha_2} = \ell\gamma\cos\alpha_1 = \pm\ell\gamma/\sqrt{2}$.

This covers all possible cases. □

By direct evaluation, we can obtain more information on the vector field near these equilibria.

Lemma 2. Equilibria in family A have a linear part with eigenvalues (with multiplicity)

$$\left\{-\frac{1}{\ell}, -\frac{1}{\ell}, \frac{1}{\ell}, \frac{1}{\ell}\right\}$$

Equilibria in family B have a linear part with eigenvalues

$$\left\{-\frac{1}{\ell}, \frac{1}{\ell}, -\frac{i}{\sqrt{2}\ell}, \frac{i}{\sqrt{2}\ell}\right\}$$

(with i denoting the imaginary unit $i^2 = -1$).

5. INTEGRABILITY PROPERTIES

Singular flow. Here, we study integrability of (17), or more precisely of its restriction to the invariant hyperplane $\{p_{\alpha_2} = 0\}$, on which the system reduces to:

$$\begin{cases} \dot{\alpha}_1 = -\dfrac{\sin\alpha_1}{\ell} - \dfrac{p_{\alpha_1}}{\gamma\ell^2} \\ \dot{\alpha}_2 = \dfrac{1}{\ell}(\sin\alpha_1 - \cos\alpha_1\sin\alpha_2) \\ \dot{p}_{\alpha_1} = \dfrac{p_{\alpha_1}}{\ell}\cos\alpha_1 \end{cases} \tag{18}$$

Theorem 3. The singular flow is not solvable by quadrature.

Proof. We remark that

$$H_r = \left(\sin(\alpha_1) + \frac{p_{\alpha_1}}{\gamma\ell}\right)^2 + \cos(\alpha_1)^2$$

is a first integral of (18). On the level $H_r = 1$, we can solve easily in p_{α_1}

$$p_{\alpha_1} = -2\sin(\alpha_1)\gamma\ell \text{ or } p_{\alpha_1} = 0.$$

This first solution gives the equation $\dot{\alpha}_1 = \frac{\sin(\alpha_1)}{\ell}$ and we deduce the following solution

$$\alpha_1(t) = 2\text{arctan}\left(e^{t/\ell}\right), \quad p_{\alpha_1}(t) = -\frac{4\gamma\ell}{e^{t/\ell} + e^{-t/\ell}}. \tag{19}$$

We will now try to obtain the corresponding solution α_2 of system (18). We introduce a variable change

$$\alpha_2(t) = i\ln z(\tau),\ t = \ell\ln\tau$$

and the equation in $z(\tau)$ becomes

$$z'(\tau) = \frac{(\tau^2-1)z(\tau)^2}{2\tau(\tau^2+1)} - \frac{2iz(\tau)}{\tau^2+1} - \frac{\tau^2-1}{2\tau(\tau^2+1)}. \tag{20}$$

This is a Riccati equation, the parameter ℓ disappears, and noting

$$z(\tau) = -\frac{2\tau(\tau^2+1)}{\tau^2-1}\frac{\phi'(\tau)}{\phi(\tau)}$$

this equation reduces to a second order linear ODE

$$\phi''(\tau) + \frac{\tau^3 + i\tau^2 - 3\tau + i}{\tau(\tau^2-1)(\tau-i)}\phi'(\tau) - \frac{(\tau^2-1)^2}{4(\tau^2+1)^2\tau^2}\phi(\tau) = 0$$

Using the Kovacic algorithm Kovacic (1986), we find that this equation has Galois group $SL_2(\mathbb{C})$, and thus is not solvable by quadrature. Thus $\alpha_2(t)$ cannot be obtained by quadrature in system (18) when α_1, p_{α_1} are given by (19).

As it is impossible to obtain $\alpha_2(t)$ by quadrature in the particular case (19), then it is not possible in general, and thus system (18) and then system (17) are not solvable by quadrature. □

Regular flow. We fix $u = -1$ ($u = 1$ is similar). In restriction to the invariant hyperplane $p_{\alpha_2} = 0$, the equations are (the time has been rescaled according to $t \to \ell t$):

$$\begin{cases} \dot{\alpha}_1 = \ell - \sin\alpha_1 \\ \dot{\alpha}_2 = \sin\alpha_1 - \cos\alpha_1\sin\alpha_2 \\ \dot{p}_{\alpha_1} = p_{\alpha_1}\cos\alpha_1 \end{cases} \tag{21}$$

Theorem 4. For $\ell \in (0, \sqrt{2}/2)$, the regular flow is not solvable by quadrature.

Proof. Again this system admits a first integral $p_{\alpha_1}(\ell - \sin\alpha_1)$, which allows to recover p_{α_1} once α_1 is known. Solving the first equation in α_1, we find the solution

$$\alpha_1(t) = 2\arctan\left(\frac{1-\sqrt{1-\ell^2}}{\ell}\tanh\left(\frac{1}{2}t\sqrt{1-\ell^2}\right)\right).$$

We substitute in the second equation, and making the variable change

$$\alpha_2(t) = i\ln(z(\tau)), \quad \tanh\left(\frac{1}{2}t\sqrt{1-\ell^2}\right)\sqrt{1-\ell^2} = \tau$$

we have a Riccati equation $z'(\tau) =$

$$\frac{(\ell^2-(\tau-1)^2)z(\tau)^2 - 4i\ell(\tau-1)z(\tau) - \ell^2 + \tau^2 - 2\tau + 1)}{(\ell^2+\tau^2-1)(\ell^2+(\tau-1)^2)}$$

We can now transform this Riccati equation to a second order linear equation by the variable change

$$z(\tau) = -\frac{(\ell^2+\tau^2-1)(\ell^2+\tau^2-2\tau+1)}{(\ell-1+\tau)(\ell+1-\tau)}\frac{\phi'(\tau)}{\phi(\tau)}$$

which gives the equation

$$\phi''(\tau) - \frac{(i\tau^4 + (\ell-3i)\tau^3 - (3i\ell^2+4\ell-3i)\tau^2 - (3\ell^3-3i\ell^2-5\ell+i)\tau - 2\ell^3 + 2\ell)}{(i\tau+\ell-i)(\ell-1+\tau)(\ell+1-\tau)(\ell^2+\tau^2-1)} \times 2\phi'(\tau) - \frac{(\ell-1+\tau)^2(\ell+1-\tau)^2\phi(\tau)}{(\ell^2+\tau^2-1)^2(\ell^2+\tau^2-2\tau+1)^2} = 0. \tag{22}$$

For a generic ℓ, the Kovacic algorithm finds no solutions, and thus generically the system (21) is not integrable by quadrature. The system could possibly nevertheless be integrated for specific values of ℓ; computing the possible confluences between singularities, we find $\ell = 0, \pm 1$ that are outside the interval studied. Thus for any $\ell \in (0, \sqrt{2}/2)$, there are exactly 6 singularities (the point at infinity is regular)

$$1-\ell, 1+\ell, 1-i\ell, 1+i\ell, \sqrt{1-\ell^2}, -\sqrt{1-\ell^2},$$

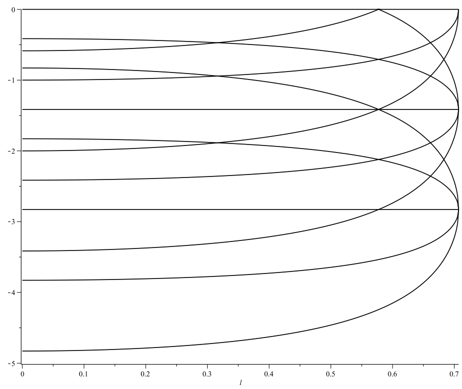

Fig. 3. The sum of exponents in function of ℓ for all possible choices of ϵ, κ. When one of these curves intersects a non positive integer ordinate, equation (22) for the corresponding ℓ has a resonance between its exponents.

with respectively the local exponents

$$(0,2),(0,2),\frac{1}{2}\pm\frac{1}{2}\sqrt{2},-\frac{1}{2}\pm\frac{1}{2}\sqrt{2},$$
$$\frac{i\ell\sqrt{1-\ell^2}\pm\sqrt{2\ell^4-3\ell^2+1}}{2(1-\ell^2)},\frac{i\ell\sqrt{1-\ell^2}\pm\sqrt{2\ell^4-3\ell^2+1}}{2(\ell^2-1)}$$

We will now follow the Kovacic algorithm Kovacic (1986) case by case. We see that not all exponents are rational, and thus not all solutions of (22) can be algebraic. Thus case III is not possible. For case I, we need a hyperexponential solution, which requires that a sum of exponents to be a non positive integer. We obtain the following equation

$$\epsilon\frac{\sqrt{2\ell^4-3\ell^2+1}}{1-\ell^2}+\kappa\sqrt{2}=-n,\ n\in\mathbb{N} \qquad (23)$$

where $\epsilon,\kappa\in\{-1,0,1\}$ depend on the choice of the exponents. This equation will give constraints to the possible ℓ. For case II, we need to consider the symmetric square of equation (22), which is a linear differential equation of order 3 with the same singularities. The computed exponents (3 for each singularity) are the following

$$(0,2,4),(0,2,4),(1,\pm\sqrt{2}),(-1,\pm\sqrt{2}),$$
$$\left(\frac{i\ell\sqrt{1-\ell^2}}{1-\ell^2},\frac{i\ell\sqrt{1-\ell^2}\pm\sqrt{2\ell^4-3\ell^2+1}}{1-\ell^2}\right),$$
$$\left(\frac{i\ell\sqrt{1-\ell^2}}{\ell^2-1},\frac{i\ell\sqrt{1-\ell^2}\pm\sqrt{2\ell^4-3\ell^2+1}}{\ell^2-1}\right).$$

We again need a hyperexponential solution, and thus a sum of exponents equal to a nonpositive integer. This gives again relation (23), but with $\epsilon,\kappa\in\{-2,-1,0,1,2\}$.

For $\ell\in(0,\sqrt{2}/2)$, we see in figure 3 that the possible non positive integer values in relation (23) are $-4,-3,-2,-1,0$, and the corresponding values of ℓ in $(0,\sqrt{2}/2)$ are

$$\tfrac{1}{3}\sqrt{3},\ \tfrac{1}{7}\sqrt{21},\ \tfrac{2}{7}\sqrt{7}-\tfrac{1}{14}\sqrt{14},\ \tfrac{2}{7}\sqrt{14}-\tfrac{1}{7}\sqrt{7},\ \tfrac{1}{2}\sqrt{3-\sqrt{2}},$$
$$\tfrac{1}{7}\sqrt{42-14\sqrt{2}},\ \tfrac{2}{21}\sqrt{84-21\sqrt{2}},\ \tfrac{1}{69}\sqrt{3933-1104\sqrt{2}},$$
$$\tfrac{1}{17}\sqrt{136\sqrt{2}-51},\ \tfrac{1}{21}\sqrt{357-168\sqrt{2}},\ \tfrac{1}{31}\sqrt{837-496\sqrt{2}}$$

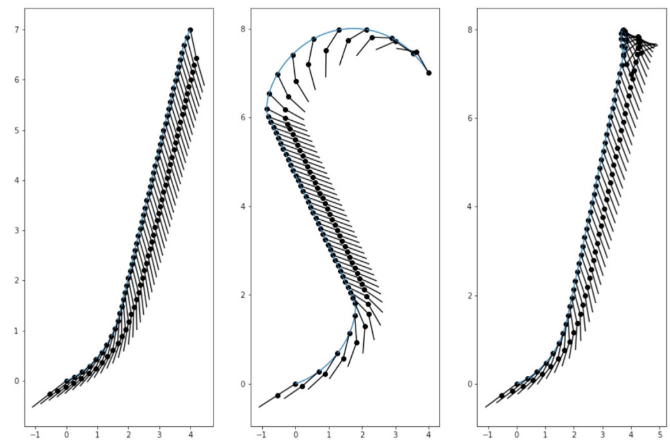

Fig. 4. Trajectories of a ship with two trailers in a constant current, unconstrained case (Zermelo-Markov-Dubins): cases (i), (ii-a) and (ii-b) from left to right. Respective minimum times are (i) $t_f=7.901$, (ii-a) $t_f=22.63$ and (ii-b) $t_f=11.38$. In particular, the extremal obtained case (ii-a) for the same problem as (ii-b) is a strict local minimum. Note that since the final trailer angles are free, no alignment is obtained. Compare with Figure 5 where the longer trajectories including two additional bang arcs allow to realign the two trailers.

For each of these distinguished values of ℓ, we apply the Kovacic algorithm and we find that none have a solvable Galois group. □

Remark. The fact that systems (18) and (21), when reduced to $p_{\alpha_2}=0$, can be solved through second order differential equations is not a generic situation, and thus further calculations could be possible through the use of special functions.

6. NUMERICAL SIMULATIONS

We use the `BOCOP` software (`python` version from the `ct` project [1]) to provide preliminary numerical computations on the problem with two trailers. The algorithm uses the midpoint scheme to discretise then optimise the problem. Three simulations are presented Figure 4 (unconstrained final trailer angles—as no alignment is required, the problem is equivalent to the standard Zermelo-Markov-Dubins one) and Figure 5 (trailers must be aligned at final time). These simulations all share the same value of the current ($w=0.8$), of ℓ ($\ell=0.6<\sqrt{2}/2$), the same initial conditions $(x_0,y_0,\theta_0)=(0,0,\pi/7)$ and final conditions $(x_f,y_f)=(4,7)$ (normalised values as explained in Section 2) while the final velocity angles differ (remember that θ is the argument of the velocity in a fixed frame, attached to the bottom of the sea): (i) $\theta=\pi/2$, (ii-a) and (ii-b) $\theta=-\pi/2$. The numerical simulations presented are reproducible online [2] on the web site of the `ct` project.

In the first case (i), one can observe that the structure of the computed trajectories is $B_+SB_-B_+B_-$ (B_+ = positive bang, that is left turn, B_- = right turn, while S = singular) *vs.* B_+SB_- for the unconstrained case. In

[1] `ct`: Control Toolbox, see ct.gitlabpages.inria.fr/gallery
[2] See ct.gitlabpages.inria.fr/gallery/nav/nav.html

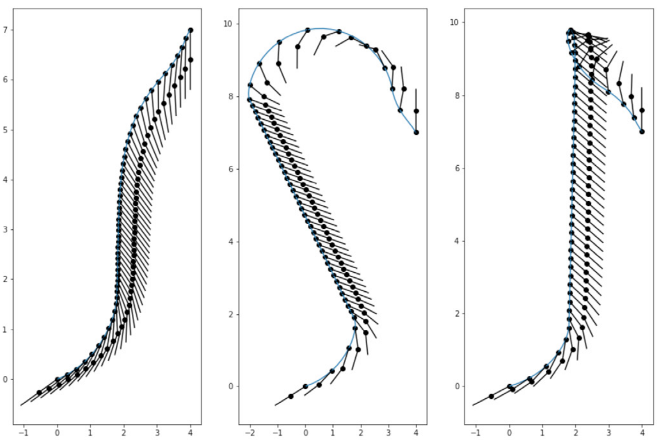

Fig. 5. Trajectories of a ship with two trailers in a constant current, constrained case (alignment of the trailers at the end): cases (i), (ii-a) and (ii-b) from left to right. Respective minimum times are (i) $t_f = 8.897$, (ii-a) $t_f = 31.82$ and (ii-b) $t_f = 19.08$. In particular, the extremal obtained case (ii-a) for the same problem as (ii-b) is a strict local minimum. Note that for case (ii-b) the curvature constraint is indeed fulfilled; as the trajectories are portrayed in a fixed frame (attached to the bottom of the sea) and not in moving frame (moving at the speed of the current, directed along the (Ox) axis), the turns in cases (ii-a) and (ii-b) apparently have different curvatures although these curvatures are indeed equal (with turns in opposite directions).

case (ii-a) the structure is again $B_+SB_-B_+B_-$ (B_+SB_- for the unconstrained case) and clearly not globally minimising, while in case (ii-b) (same problem, but different solution) the structure is $B_+SB_+B_-B_+$ (B_+SB_+ in the unconstrained case) and gives a much better final time. In the three cases, as for one trailer Caillau et al. (2019) the main part of the trajectory is a singular arc. This illustrates the so-called turnpike phenomenon; for a distant enough target in the (x, y) plane, the requested minimum time is long enough to allow the singular arc of the extremal to come close to the hyperbolic equilibrium $(0, 0, 0, 0)$ of family A described in Section 4. The longer the minimum time, the closer this singular part is to the straight line encountered in the Zermelo-Markov-Dubins problem without trailer. In contrast with the one trailer case, the trajectories are terminated not by two but three bang arcs that accommodate the alignment requirement of the two trailers at the end. These short bang arcs are much more time efficient than the "run-in" procedure (a final straight line) performed until now by ships at the end of the maneuver during real exploration campaigns. Such comparisons will be the topic of future work.

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