Problem 3.1

Minimum time control of the Kepler equation

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1 DESCRIPTION OF THE PROBLEM

We consider the controlled Kepler equation in three dimensions

$$\ddot{r} = -k\frac{r}{|r|^3} + \gamma \tag{1}$$

where $r = (r_1, r_2, r_3)$ is the position vector–the double dot denoting the second order time derivative–, k a strictly positive constant, $|.|$ the Euclidean norm in $\mathbf{R}^3$, and where $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ is the control. The minimum time problem is then stated as follows: find a positive time T and a measurable function γ defined on $[0, T]$ such that (1) holds almost everywhere on $[0, T]$ and:

$$T \to \min$$

$$r(0) = r^0,\ \dot{r}(0) = \dot{r}^0 \tag{2}$$

$$h(r(T), \dot{r}(T)) = 0 \tag{3}$$

$$|\gamma| \leq \Gamma. \tag{4}$$

In (2), r^0 and $\dot{r}^0$ are the known initial position and speed with:

$$\frac{|\dot{r}^0|^2}{2} - \frac{k}{|r^0|} < 0$$

in order that the uncontrolled initial motion be periodic [1]. In (3) h is a fixed submersion of $\mathbf{R}^6$ onto $\mathbf{R}^l$, $l \leq 6$, defining a non-trivial endpoint condition. The constraint (4) on the Euclidean norm of the control, with Γ a strictly positive constant, means that almost everywhere on $[0, T]$

$$\gamma_1^2 + \gamma_2^2 + \gamma_3^2 \leq \Gamma^2.$$

Our first concern is uniqueness (see §3 about existence):

Question 1. *Is the optimal control unique?*

The second point is about regularity, namely:

Question 2. *Are there continuous optimal controls?*

Denoting by $T(\Gamma)$ the value function that assigns to any strictly positive Γ (parameter involved in (4)) the associated minimum time, our third and last question is:

Question 3. *Does the product* $T(\Gamma) \cdot \Gamma$ *have a limit when* Γ *tends to zero?*

2 MOTIVATION

This problem originates in the computation of optimal orbit transfers in celestial mechanics for satellites with very low thrust engines [5]. Since the 1990s, low electro-ionic propulsion is been considered as an alternative to strong chemical propulsion, but the lower the thrust, the longer the transfer time, hence the idea of minimizing the final time. In this context, γ is the ratio u/m of the engine thrust by the mass of the satellite, and one has moreover to take into account the mass variation due to fuel consumption:

$$\dot{m} = -\beta|u|.$$

Typical boundary conditions in this case consist in inserting the spacecraft on a high geostationnary orbit, and the terminal condition is defined by:

$$|r(T)| \text{ and } |\dot{r}(T)| \text{ fixed, } r(T) \cdot \dot{r}(T) = 0, \; r(T) \times \dot{r}(T) \times \vec{k} = 0$$

where $\vec{k}$ is the normal vector to the equatorial plane.

In contrast with the impulsional manoeuvres performed using the strong classic chemical propulsion, the gradual control by a low thrust engine is sometimes referred to as "continuous?" Thus, question 2 could be rephrased according to:

Are "continuous" optimal controls continuous?

Besides, this question is also relevant in practice since continuity of controls is the basic assumption required by most numerical methods [2]. In the same respect, question 3 is the key to get accurate estimates of the unknown transfer time, needed to ensure convergence of the numerical computation.

3 RELATED RESULTS

The existence of controls achieving the minimum time transfer comes from the controllability of the system (the associated Lie algebra has maximal rank and the drift is periodic, see [7]) and from the convexity properties of the dynamics by Filippov theorem [4]. Regarding regularity, it is proven in [3] (whose results extend straightforwardly to three dimensions) that any time minimal control of (1) has at most finitely many discontinuity points. More precisely, using Pontryagin Maximum Principle [4, 6], one gets that any discontinuity point $\bar{t}$ is a switching point in the sense that the control is instantaneously rotated of an angle π:

$$\gamma(\bar{t}+) = -\gamma(\bar{t}-).$$

Furthermore, bounds are given in [3], not on the total number of switchings but for those located at special points of the osculating ellipse: there cannot be consecutive switchings at perigee or apogee. Since the numerics suggest that the possible discontinuities are exactly located at the perigee, a conjecture would be:

There is at most one switching point and this point is located at the perigee.

Finally, as for question 3, the value function $T(\Gamma)$ is obviously decreasing and is proven to be right-continuous in [2]. Besides, the product $T(\Gamma) \cdot \Gamma$ turns to be nearly constant numerically so that the conjecture would be to answer positively:

> *There is a positive constant c such that $T(\Gamma) \cdot \Gamma$ tends to c when Γ tends to zero.*

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