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SENSITIVITY ANALYSIS FOR TIME OPTIMAL ORBIT TRANSFER

JEAN-BAPTISTE CAILLAU* and JOSEPH NOAILLES

ENSEEIHT-IRIT, UMR CNRS 5505, 2 rue Camichel, F-31071 Toulouse, France

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The minimum time transfer of a satellite around the Earth is studied. In order to deal numerically with low thrusts, a new method is introduced: Based on a so-called non-controllability function, the technique treats the final time as a parameter. The properties of the method are studied by means of an infinite dimensional sensitivity analysis. The numerical results obtained by this approach for very low thrusts are given.

Keywords: Low thrust orbit transfer; Time optimal control; Parametric optimization

Mathematics Subject Classifications 1991: Primary: 70Q05; Secondary: 49K40, 49M30

1. INTRODUCTION

We consider the transfer of a satellite from a low initial orbit around the Earth towards a high geostationnary terminal one. The transfer treated is coplanar, and the performance index to be minimized is the total transfer time. The satellite is described by its state, that is for instance the couple position-speed (we shall not take into account the variation of the mass). The control applied to the spacecraft is the thrust of its engine. As we are concerned with new-generation engines (electroionic ones), the thrusts are supposed to be very weak (*e.g.*, 0.3 Newton for a 1.5 Ton spaceship). As a consequence, the resulting transfer times are very long (up to four months for 0.3 Newton).

*Corresponding author.

Moreover, since the initial elliptic orbit is very eccentric, the dynamics is strongly nonlinear. For these reasons, the associated optimal control problem is numerically hard to solve with classical direct or (semi-) indirect methods [3]. As in [11], the idea is then to tackle the difficulties separately and to treat the criterion, here the transfer time, specifically: Given a fixed terminal time smaller than the optimal one, we measure how close to the aim (the geostationnary orbit) we are able to get. This gives us a measurement of the non-controllability of the system with respect to the terminal condition for a prescribed transfer time. Obviously, the optimal time is then the first instant such that this measurement is zero. Furthermore, dealing with the criterion separately increases the robustness of the resolution as demonstrated by the numerical experiments.

We begin by recalling the optimal control problem in Section 2. Some preliminary results of existence and regularity of the control are stated. Then we define the non-controllability function and the method in Section 3. Lipschitz continuity of the function is proved under quite general assumptions. In Section 4, we go back to our transfer problem and perform a sensitivity analysis with the tools of [10] in order to show continuous differentiability of the process. Some of the assumptions used are only verifiable numerically, which is done in the Section 5, devoted to the numerical experiments. In particular, the optimal times, trajectories and controls are given for very low thrusts.

2. TIME OPTIMAL ORBIT TRANSFER

We first recall the optimal control formulation of the transfer problem. To this end, rather than using the canonical cartesian coordinates for the state (position and speed of the satellite in a fixed geocentric frame), we choose the orbital parameters of the osculating ellipse to the trajectory:

$$x = (P, e_x, e_y, L)$$

where P is the semi-latus rectum, $e = (e_x, e_y)$ the eccentricity vector, and L the true longitude (see Fig. 1). The control is then expressed in the moving frame attached to the satellite forming an angle L with the

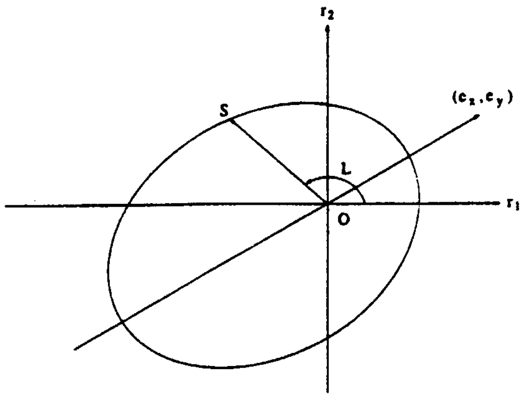

FIGURE 1 Orbital coordinates of satellite S defining an ellipse of focus O (center of the Earth), semilatus rectum P, eccentricity vector $e=(e_x, e_y)$, and true longitude L. The axes (r_1, r_2) define a fixed geocentric frame.

fixed geocentric frame. The dynamics, defined on a suitable open submanifold of $\mathbf{R}^n$ $(n=4)$, is

$$\dot{x} = f_0(x) + u_1 f_1(x) + u_2 f_2(x) \tag{1}$$

In the previous systems of coordinates, the vectors fields are

$$f_0 = \sqrt{\mu^0/P}\begin{bmatrix} 0 \\ 0 \\ 0 \\ W^2/P \end{bmatrix} \tag{2}$$

$$f_1 = \sqrt{P/\mu^0}\begin{bmatrix} 0 \\ \sin L \\ -\cos L \\ 0 \end{bmatrix} \tag{3}$$

$$f_2 = \sqrt{P/\mu^0}\begin{bmatrix} 2P/W \\ \cos L + (e_x + \cos L)/W \\ \sin L + (e_y + \sin L)/W \\ 0 \end{bmatrix} \tag{4}$$

$$W = 1 + e_x \cos L + e_y \sin L$$

where μ^0 is the gravitation constant of the Earth. We will also use the matrix $B \in \mathcal{L}(\mathbf{R}^m \cdot \mathbf{R}^n)$ (the dimension m of the control being 2 since the transfer is coplanar)

$$B = [f_1\, f_2]$$

Hence, our problem is to find the smallest positive transfer time t_f

$$t_f \to \min$$

together with an absolutely continuous trajectory x in the space $W_n^{1,\infty}([0,t_f]) = W_n^{1,\infty}([0,t_f],\mathbf{R}^n)$, and a measurable control u in the space of essentially bounded functions $L_m^\infty([0,t_f]) = L^\infty([0,t_f],\mathbf{R}^m)$ such that the dynamics (1) plus the following constraints are verified on $[0,t_f]$:

$$x(0) = x^0, \quad h(x(t_f)) = 0 \tag{5}$$

$$|u| \le \gamma_{\max} \tag{6}$$

$$(t,x) \in A \tag{7}$$

Equation (5) defines the boundary constraints, in particular the insertion on the final orbit:

$$h(P, e_x, e_y, L) = \begin{bmatrix} P - P^f \\ e_x - e_x^f \\ e_y - e_y^f \end{bmatrix}$$

Equation (6) gives the upper bound on the thrust, that is the maximum acceleration $\gamma_{\max}$ we can use. The norm $|\cdot|$ is the Euclidean norm issued from the usual scalar product in $\mathbf{R}^m$, so that (6) is equivalent to:

$$u_1^2 + u_2^2 \le \gamma_{\max}^2$$

We shall use the same convention in the rest of the paper that, when unspecified, norms on finite dimensional spaces are Euclidean. As mentioned in the Introduction, $\gamma_{\max}$ is very small in practice. The last Eq. (7) defines a path constraint meant to ensure the satellite does not collide with the Earth and that the trajectory remains elliptic:

$$P \ge \Pi^0 > 0$$
$$|e| \le \varepsilon^0 < 1$$

The transfer problem will be referred to as $(\mathrm{SP})_{\gamma_{\max}}$.

We now mention some facts on controllability and smoothness of the optimal control that will be useful for our study. Taking advantage of the particular structure of the Lie algebra generated by the vectors fields (2)–(4) defining the dynamics, it is proven in [5] that the system is controllable for any strictly positive constraint on the modulus of the control. Therefore, we have

PROPOSITION 1 *For any strictly positive γ_{max}, there exists an optimal control realizing the minimum time transfer.*

Proof Since we know the system is controllable, the set of admissible trajectories is not empty; We can thus find $T > 0$ such that the system is controllable in time T and restrict the path constraint to $A \cap ([0, T] \times \mathbf{R}^n)$. Besides, since

$$\begin{aligned} W &= 1 + e_x \cos L + e_y \sin L \\ &\geq 1 - \varepsilon^0 > 0 \end{aligned}$$

there is a positive constant C such that we can bound the first component of the dynamics by a Lyapunov function V:

$$\begin{aligned} V'(P)\left(2\sqrt{P^3/\mu^0}u_2/W\right) &\leq V'(P)C\sqrt{P^3} \\ &\leq CV(P) \end{aligned}$$

with $V(P) = \exp(-2/\sqrt{P})$, $P \geq \Pi^0 > 0$. We can write similar bounds for the other components of f and conclude that the admissible trajectories stay in a fix compact subset of $\mathbf{R}^{1+n}$ included in A. At last, the control set which is a closed ball in the Euclidean space $\mathbf{R}^m$ is compact and convex, the dynamics being also convex: The existence result follows by Filippov theorem [6]. ■

Concerning the structure of the optimal command, it can be proved that it is smooth[1] everywhere excepted in isolated points, called *commutation* or *switching* points. The main geometric results of [4] are summarized in

PROPOSITION 2 *There is a finite number of commutations, and if $t_i < t_{i+1}$ are two consecutive switching instants, the optimal control and the optimal trajectory are smooth on $[t_i, t_{i+1}]$. Moreover, there cannot be consecutive commutations at perigee or apogee.*

[1] As usual, by *smooth* we mean indefinitely differentiable.

As we shall see in Section 5, the numerical simulation shows that the possible commutations occur precisely at the perigee of the osculating ellipse, so there may be at most one switching. In practice, the basic assumption on the problem is that there is no switching at all [3]. We now go into details of the method.

3. NON-CONTROLLABILITY FUNCTION

Let (OCP) be a general time optimal control problem

$$t_f \rightarrow \min$$

with dynamics

$$\dot{x} = f(t, x, u)$$

end-point, control and path constraints

$$\begin{aligned} &x(0) = x^0, \quad h(x(t_f)) = 0 \\ &u \in U(t, x) \\ &(t, x) \in A \end{aligned}$$

We suppose that the functions f and h are smooth on appropriate open subsets of $\mathbf{R}^{1+n+m}$ and $\mathbf{R}^n$ respectively, h being a submersion onto $\mathbf{R}^l$. We also impose that A and

$$N = \{(t, x, u) \in \mathbf{R}^{1+n+m} | (t, x) \in A \text{ and } u \in U(t, x)\}$$

be closed. The optimal trajectory is sought as an absolutely continuous function x in $W_n^{1,\infty}([0, t_f])$, the control as an essentially bounded function u in $L_m^\infty([0, t_f])$. To (OCP) we associate the parametric family $(\text{OCP})_\beta$ of optimal control problems with fixed final time $\beta \geq 0$ and without end-point constraint,

$$\begin{aligned} &1/2|h(x(\beta))|^2 \rightarrow \min \\ &\dot{x} = f(t, x, u) \\ &x(0) = x^0 \\ &u \in U(t, x) \\ &(t, x) \in A \end{aligned}$$

and define

DEFINITION 1 The non-controllability function ϕ is the value function of the family $(\mathrm{OCP})_\beta$ that maps $\beta \in \mathbf{R}_+$ to $\phi(\beta) \in \bar{\mathbf{R}}$, the optimal value of $(\mathrm{OCP})_\beta$.

Then, defining the problem (E) of finding the first zero of ϕ, we have

PROPOSITION 3 *(OCP) and (E) are equivalent in the sense that any solution of one of them defines a solution of the other, (E) having at most one solution.*

Proof Let $\bar{\beta}$ be the solution of (E), and let $(\bar{x}, \bar{u})$ be the associated admissible couple for $(\mathrm{OCP})_{\bar{\beta}}$. Then $(\bar{\beta}, \bar{x}, \bar{u})$ is obviously solution of (OCP) since otherwise, one could find an optimal time $\tilde{\beta} < \bar{\beta}$ contradicting the fact that $\bar{\beta}$ is the smallest zero of ϕ. The converse is also trivial. ∎

The theoretical asset of the method lies in the explicit management of the criterion. In particular, if (OCP) has local minima (and thus non-optimal points satisfying first order necessary conditions), all of these will be zeros of ϕ strictly greater than the solution $\bar{\beta}$ (if it exists). Now, if we use a Newton-like algorithm (assuming ϕ differentiable) to solve the scalar equation (E')

$$\phi(\beta) = 0$$

if the nonlinear resolution is initialized with $\beta_0 < \bar{\beta}$, the iterates β_k produced by the algorithm will converge to $\bar{\beta}$ without exceeding it provided ϕ has some convexity properties. On the opposite, classical methods in optimal control such as single shooting do not treat separately the performance index, especially in the minimum time problems where the final time is made a parameter or a state variable. Hence, there is no specific treatment of its value during the iterative process, and local minima are often encountered. Furthermore, this uncoupling of the criterion from the others unknowns proves to be numerically relevant, as will be demonstrated in Section 5.

Remark 1 In [11], a similar approach is introduced for time optimal problems. The auxiliary problems are also with fixed terminal time, but the non-controllability is measured with respect not to the end-point constraint but to the dynamics: For a prescribed final time smaller than the optimal one, a new control is added to the dynamics so as to make the system controllable. The performance index of the auxiliary problems is then the L^2 norm of this *fictitious* control. The main drawback of that method is that it happens to alter too much the dynamics: When away from the solution, a strong additional control is required to reach the end-point constraint. This disadvantage is much less decisive with our method whose simpler auxiliary problems $(\mathrm{OCP})_\beta$ do not include the end-point constraint.

We shall need the assumptions hereafter on (OCP):

(I3.1) the set of admissible triples (t_f, x, u) for (OCP) is non-empty;
(I3.2) the state-control set N is compact;
(I3.3) $Q(t,x) = f(t, U(t,x))$ is convex, $(t,x) \in A$;
(I3.4) for any $(t_1, x_1) \in A$, $t_2 > t_1$, there is a trajectory x in $W_n^{1,\infty}([t_1, t_2])$ and a control u in $L_m^\infty([t_1, t_2])$ such that on $[t_1, t_2]$:

$$\begin{aligned}
&\dot{x} = f(t, x, u),\\
&x(t_1) = x_1, \ \ h(x(t_2)) = h(x(t_1))\\
&u \in U(t,x),\\
&(t,x) \in A
\end{aligned}$$

This last assumption means that, from any point in A, it is possible to reach a later time by means of an admissible trajectory *without changing the value of the end-point constraint.* As stated in the end of this section, this assumption is straightforwardly fulfilled on the transfer problem.

PROPOSITION 4 *Under assumptions (I3.1)–(I3.4), ϕ is finite and decreasing on $\boldsymbol{R}_+$. Besides, $\phi(\beta) = 0$ for any β greater than its first zero $\bar{\beta}$.*

Proof The previous assumptions allow us to apply Filippov theorem to (OCP) which thus has a solution $(\bar{t}_f, \bar{x}, \bar{u})$; Hence, $\bar{\beta} = \bar{t}_f$ is the first zero of ϕ. If $\beta < \bar{\beta}$, the restriction of $(\bar{x}, \bar{u})$ to $[0, \beta]$ is admissible for $(\mathrm{OCP})_\beta$ which in turn has also a solution: $\phi(\beta)$ is finite. Finally, ϕ is obviously decreasing by virtue of (I3.4). Indeed, if $\phi(\beta_1)$ is finite, for any $\beta_2 > \beta_1$, we can extend the couple (x_1, u_1) associated with β_1 to $[0, \beta_2]$ by means of an admissible trajectory such that the resulting pair (x_2, u_2) be admissible for $(\mathrm{OCP})_{\beta_2}$ and verify $h(x_2(\beta_2)) = h(x_1(\beta_1))$: $\phi(\beta_2)$ is finite and less than $\phi(\beta_1)$. Since ϕ is also positive, we conclude that it is necessarily zero after $\bar{\beta}$. ■

Under the same conditions, we have a first result on the regularity of ϕ:

PROPOSITION 5 *Under assumptions* $(I3.1)-(I3.4)$, ϕ *is lipschitz continuous.*

Proof Let β_1 and β_2 be such that $0 \leq \beta_1 \leq \beta_2 \leq \bar{\beta}$; For any $\beta \in [0, \bar{\beta}]$, let $(x(\cdot, \beta), u(\cdot, \beta))$ be a solution of $(\mathrm{OCP})_\beta$. Almost everywhere on $[0, \beta]$

$$\dot{x}(t, \beta) = f(t, x(t, \beta), u(t, \beta))$$

f is continuous and N is compact, so the family $(x(\cdot, \beta))_\beta$ is equilipschitzian on $[0, \bar{\beta}]$ ($x(\cdot, \beta)$ is extended on $[0, \bar{\beta}]$ by constancy and continuity). As the trajectories remain in a fix compact (N is compact, thus A also), the family $((1/2)|h(x(\cdot, \beta))|^2)_\beta$ is still equilipschitzian because h is smooth, *i.e.*, we can find a positive constant k independent of β such that:

$$\left| \frac{1}{2}|h(x(t_1, \beta))|^2 - \frac{1}{2}|h(x(t_2, \beta))|^2 \right| \leq k|t_1 - t_2|, \quad (t_1, t_2) \in [0, \bar{\beta}]^2$$

Now, since ϕ is decreasing by Proposition 4, we have

$$0 \leq \phi(\beta_1) - \phi(\beta_2) \leq \frac{1}{2}|h(x(\beta_1, \beta_2))|^2 - \frac{1}{2}|h(x(\beta_2, \beta_2))|^2 \qquad (8)$$

for the restriction of $(x(\cdot, \beta_2), u(\cdot, \beta_2))$ to $[0, \beta_1]$ is admissible for $(\mathrm{OCP})_{\beta_1}$. We get from (8) that

$$|\phi(\beta_1) - \phi(\beta_2)| \leq k|\beta_1 - \beta_2|$$

and the proof is finished. ■

Remark 2 In general it is not true that $\phi(\beta) = 1/2|h(\bar{x}(\beta))|^2$ (where $\bar{x}$ is a solution of (OCP)). However, the inequality

$$\phi(\beta) \leq 1/2|h(\bar{x}(\beta))|^2, \quad \beta \in [0, \bar{\beta}] \qquad (9)$$

may permit to detect whether the numerical evaluation has produced an upper estimation of ϕ (*cf.* Section 5).

The next statement examines the behaviour of ϕ in the neighbourhood of $\bar{\beta}$. Assuming that

(I3.5) there exists an optimal trajectory of (OCP) two times differentiable at $\bar{t}_f$;
we have

PROPOSITION 6 *Under assumptions (I3.1)–(I3.5), ϕ is differentiable at $\bar{\beta}$ with $\phi'(\bar{\beta}) = 0$, and $\phi(\beta) = O((\beta - \bar{\beta})^2)$ in the neighbourhood of $\bar{\beta}$.*

Proof Let β be such that $0 \leq \beta < \bar{\beta}$; then

$$0 \leq \frac{\phi(\beta) - \phi(\bar{\beta})}{(\beta - \bar{\beta})^2} \leq \frac{1/2|h(\bar{x}(\beta))|^2}{(\beta - \bar{\beta})^2}$$

Now, (I3.5) implies that $\psi(\beta) = 1/2|h(\bar{x}(\beta))|^2$ is two times differentiable at $\bar{\beta}$, so

$$\frac{1/2|h(\bar{x}(\beta))|^2}{(\beta - \bar{\beta})^2} \to 1/2\psi''(\bar{\beta})$$

when $\beta \to \bar{\beta}$ since $\psi(\bar{\beta}) = \psi'(\bar{\beta}) = 0$ (*cf.* $h(\bar{x}(\bar{t}_f)) = 0$). ∎

Remark 3 When an iterative process is used to find a zero of ϕ, a sequence $(\beta_k)_k$ converging to $\bar{\beta}$ is generated. It is then desirable that, if for each $k(x_k, u_k)$ is a solution of $(\mathrm{OCP})_{\beta_k}$, the last sequence tends in a suitable way to a solution $(\bar{x}, \bar{u})$ of (OCP). In fact, it is proven in [5] that, under mild assumptions, a subsequence of the latter converges to a solution couple, uniformly in the state and weakly-$*$ in the control:

$$\begin{aligned} x_k &\to \bar{x}, \quad k \to \infty, \quad \text{in } (C_n^0([0, \bar{\beta}]), \|\cdot\|_\infty) \\ u_k &\to \bar{u}, \quad k \to \infty, \quad \text{in } (L_m^\infty([0, \bar{\beta}]), \sigma(L^\infty, L^1)) \end{aligned}$$

In order to apply these statements to $(\mathrm{SP})_{\gamma_{\max}}$, it is enough to check assumption (I3.4). Indeed, we already mentioned in Section 2 that the system is controllable, and though in our case A (and hence N) is not bounded, we showed that the admissible trajectories remain in a fix compact: As a consequence, the state-control space N can be restricted to a given compact subset of $\mathbf{R}^{1+n+m}$ and condition (I3.2) is true. Assumption (I3.5) also holds by virtue of Proposition 2. Now, if (t_1, x_1) belongs to A, it is always possible to steer the system to a later time t_2 without changing the end-point value by using an admissible control between t_1 and t_2, namely the zero control: the systems oscillates freely on its current orbit, and none of the components of the state are changed but the true longitude L. Since it is not constrained, our statement is true. Therefore, we know that the non-controllability function associated with the transfer problem is almost everywhere differentiable and has a quadratic behaviour in the neighbourhood of the optimal transfer time. The purpose of next section is to define extra assumptions ensuring continuous differentiability.

4. SENSITIVITY ANALYSIS

Our aim is to treat the transfer problem numerically using the method described in the previous section. More precisely, the idea consists in solving the nonlinear scalar equation (E')

$$\phi(\beta) = 0$$

starting with an initial approximation β_0 smaller than the first zero. In order to apply a Newton-like algorithm, we need more than lipschitz continuity of ϕ, namely continuous differentiability. To this end, we perform a sensitivity analysis on the parametric optimal control family associated with the transfer problem, using the tools of [9, 10]. As in finite dimensional optimization, the first step is to construct an extremal family satisfying KKT conditions thanks to the implicit function theorem. Then, a second order sufficient condition is called upon to ensure optimality (at least locally) of the former extremals. The conditions needed are of course quite similar to those in finite dimension (regularity of multipliers, strict complementarity...), with the same overlap of coercivity over regularity. But beyond the peculiarities induced by the control setting (Legendre–Clebsch or Jacobi conditions, for instance), the essential characteristic of the infinite dimensional framework is the so-called 'two-norm discrepancy' [10]. Indeed, whereas the problem is naturally topologized in a suitable Banach space, the coercivity condition is expressed in the weaker topology (strictly weaker because of the infinite dimension) of the Hilbert space into which the former is embedded. Over and above, among the conditions required for sensitivity, some are only verifiable numerically.

For an arbitrary maximum acceleration $\gamma_{\max} > 0$, the non-controllability function is the value function of the parametric optimal control problem $(\mathrm{SP})^{\beta}_{\gamma_{\max}}$

$$\begin{aligned} &1/2|h(x(1))|^2 \to \min \\ &\dot{x} = \beta(f_0(x) + B(x)u) \\ &x(0) = x^0 \\ &|u| \leq \gamma_{\max} \end{aligned}$$

where we have recast the problem on [0, 1] (so the parameter appears explicitly in the dynamics) and omitted the path constraint according to

(I4.1) any optimal trajectory of $(\mathrm{SP})_{\gamma_{\max}}$ is interior to A.

Let then β_0 belong to $]0, \bar{\beta}[$; $(\mathrm{SP})^{\beta_0}_{\gamma_{\max}}$ has a solution (x_0, u_0) in $W^{1,\infty}_n([0,1]) \times L^{\infty}_m([0,1])$ such that the maximum principle applies: There are Lagrange multipliers (p_0, ν_0) in $W^{1,\infty}_n([0,1]) \times \mathbf{R}^n$ such that, almost everywhere on [0, 1],

$$\dot{p}_0 = -\nabla_x H(x_0, u_0, p_0, \beta_0) \tag{10}$$

$$p_0(0) = \nu_0, \quad p_0(1) = {}^t h'(x_0(1)) h(x_0(1)) \tag{11}$$

$$H(x_0, u_0, p_0, \beta_0) = \min_{|v| \leq \gamma_{\max}} H(x_0, v, p_0, \beta_0) \tag{12}$$

with hamiltonian $H(x, u, p, \beta) = \beta(p|f_0(x) + B(x)u)$ (where $(\cdot|\cdot)$ is the Euclidean dot product). The first order necessary condition is here in qualified form since otherwise the adjoint state p would be identically zero which is impossible (as $\beta_0 < \bar{\beta}$, $h(x_0(1)) \neq 0$ and the differential of the criterion is not zero because h is a submersion; This ensures that the adjoint state never vanishes). Appealing again to the geometric arguments of [4], one can prove that ${}^t B(x_0) p_0$ has a finite number of zeros on [0, 1]. Apart from these possible switching points, by virtue of (10)–(12) the optimal control is given by

$$u_0 = -\gamma_{\max} \frac{{}^t B(x_0) p_0}{|{}^t B(x_0) p_0|} \tag{13}$$

We shall observe in Section 5 that, in accordance with Proposition 2, the optimal control of $(\mathrm{SP})_{\gamma_{\max}}$ has at most one switching located at the perigee. Nevertheless, it is compulsory to suppose more on the auxiliary problems, namely that, for each β in $]0,\bar{\beta}[$, $(\mathrm{SP})^{\beta}_{\gamma_{\max}}$ has a solution $(x(\cdot,\beta),u(\cdot,\beta))$ such that

(I4.2) $u(\cdot,\beta)$ is continuous.

Hence the control is smooth and x_0 and p_0 are solution of the two-point boundary value problem $(\mathrm{BVP})_{\beta_0}$

$$\dot{x} = \beta_0(f_0(x) + B(x)u(x,p)) \tag{14}$$

$$\dot{p} = -\nabla_x H(x,u(x,p),p,\beta) \tag{15}$$

$$x(0) = x^0, \quad p(1) = {}^t h'(x(1))h(x(1)) \tag{16}$$

$$u(x,p) = -\gamma_{\max}{}^t B(x)p/|{}^t B(x)p| \tag{17}$$

with u defined as a mapping of (x,p) by (17). One can then find an open neighbourhood V of $(p_0(0),\beta_0)$ such that the smooth maximal flow [2] $\varphi(t,y,\beta)$ of the initial value problem $\dot{y} = \xi(y,\beta)$, where ξ denotes the second member of (14)–(15) with $y=(x,p)$, is defined on an open subset containing $[0,1]\times\{x^0\}\times V$. On this neighbourhood V, we thus define

$$S(p^0,\beta) = b(\varphi(1,(x^0,p^0),\beta))$$

($b(y)=p-{}^t h'(x)h(x)$ is the boundary function associated with (16)). For a fixed β, $S(\cdot,\beta)$ is the *shooting function* and $(\mathrm{BVP})_\beta$ is equivalent to the shooting equation: Find p^0 such that

$$S(p^0,\beta) = 0 \tag{18}$$

The authors of [10] then apply the implicit function theorem to (18) to construct an extremal family of $(\mathrm{SP})^{\beta}_{\gamma_{\max}}$ in a neighbourhood of β_0. At last, a coercivity condition in the form of a Riccati equation ensures (local) optimality of these extremals. Accordingly, so as to proceed in the same way for arbitrary β in $]0,\bar{\beta}[$, we must suppose that $(\mathrm{SP})_{\gamma_{\max}}$ has a solution $(x(\cdot,\beta),u(\cdot,\beta))$ together with multipliers $(p(\cdot,\beta),\nu(\beta))$ such that the following regularity and second order sufficient conditions hold:

(I4.3) $\partial_p S(p(0,\beta),\beta)$ belongs to $\mathrm{GL}_n(\mathbf{R})$;

(I4.4) the symmetric Riccati equation hereafter (see [10] for more details) has a solution ($\mathbf{I}_3$ denotes the identity matrix of order 3)

$$\dot{Q} = -Q\mathcal{A}(t,\beta) - {}^t\mathcal{A}(t,\beta)Q + Q\mathcal{B}(t,\beta)Q - \mathcal{C}(t,\beta) \tag{19}$$

$$((R^f - Q(1))v|v) \geq 0, \quad v \in \mathbf{R}^4 \tag{20}$$

$$\begin{aligned}
\mathcal{A}(t,\beta) &= \partial_x \xi_1(x(t,\beta), p(t,\beta), \beta) \\
\mathcal{B}(t,\beta) &= \partial_p \xi_1(x(t,\beta), p(t,\beta), \beta) \\
\mathcal{C}(t,\beta) &= \partial_x \xi_2(x(t,\beta), p(t,\beta), \beta) \\
R^f &= \begin{bmatrix} \mathbf{I}_3 & 0 \\ 0 & 0 \end{bmatrix}
\end{aligned}$$

These last assumptions will be checked numerically in next section. We are now able to state our last result:

PROPOSITION 7 *Under assumptions* (*I*4.1)–(*I*4.4), ϕ *is continuously differentiable on* $]0, \bar{\beta}[$ *and*

$$\phi'(\beta) = H(0,\beta)/\beta \tag{21}$$

In so far as (21) is only expressed in terms of the hamiltonian of $(\mathrm{SP})^{\beta}_{\gamma_{\max}}$ (constant along the optimal trajectory), the computation of ϕ' does not make use of the variational derivatives $\partial_\beta x(\cdot,\beta)$, $\partial_\beta u(\cdot,\beta)$ or $\partial_\beta p(\cdot,\beta)$, and is therefore straightforward. We first prove a lemma on the abstract parametric optimization problem with equality constraints $(O)_\beta$ (the parameter is still denoted β)

$$\begin{aligned}
J(z,\beta) &\to \min \\
F(z,\beta) &= 0
\end{aligned}$$

with $J : \mathcal{Z} \times \mathcal{B} \to \mathbf{R}$ and $F : \mathcal{Z} \times \mathcal{B} \to \mathcal{Y}$ differentiable, $\mathcal{Z}$, $\mathcal{B}$ and $\mathcal{Y}$ Banach spaces. Let $W(\beta)$ be the value function of $(O)_\beta$, let the lagrangian (in qualified form) of the problem be

$$L(z,\lambda,\beta) = J(z,\beta) + \langle \lambda, F(z,\beta) \rangle_{\mathcal{Y}',\mathcal{Y}}, \quad (z,\lambda,\beta) \in \mathcal{Z} \times \mathcal{Y}' \times \mathcal{B}$$

where $\langle \cdot,\cdot \rangle_{\mathcal{Y}',\mathcal{Y}}$ is the duality pairing between $\mathcal{Y}$ and its dual. If for each β in an open neighbourhood V of β_0 the problem $(O)_\beta$ has a solution $z(\beta)$ in $\mathcal{Z}$ and an associated multiplier $\lambda(\beta)$ in $\mathcal{Y}'$ verifying KKT conditions such that the mappings $\beta \mapsto z(\beta)$ and $\beta \mapsto \lambda(\beta)$ are differentiable, we have (more generally, see [8])

LEMMA 1 *W is differentiable on V and*

$$W'(\beta) = \partial_\beta J(z(\beta), \beta) + \lambda(\beta)\partial_\beta F(z(\beta), \beta)$$

Proof Since $W(\beta) = J(z(\beta), \beta)$, W is differentiable on V and

$$W'(\beta) = \partial_z J(z(\beta), \beta) z'(\beta) + \partial_\beta J(z(\beta), \beta)$$

As $\partial_z L(z(\beta), \lambda(\beta), \beta) = 0$ in V (KKT conditions)

$$\partial_z J(z(\beta), \beta) + \lambda(\beta)\partial_z F(z(\beta), \beta) = 0$$

and thus

$$W'(\beta) = \partial_\beta J(z(\beta), \beta) - \lambda(\beta)\partial_z F(z(\beta), \beta) z'(\beta)$$

Besides, because $F(z(\beta), \beta) = 0$, $\beta \in V$, one has

$$\partial_z F(z(\beta), \beta) z'(\beta) + \partial_\beta F(z(\beta), \beta) = 0$$

that is

$$W'(\beta) = \partial_\beta J(z(\beta), \beta) + \lambda(\beta)\partial_\beta F(z(\beta), \beta)$$

which concludes the proof. ∎

Proof (of Proposition 7) So as to apply Lemma 1 to ϕ, it is enough to check the assumptions of the sensitivity analysis result of [10] (Theorem 3, p. 274). Let us take an arbitrary β_0 in $]0, \bar{\beta}[$; $(\mathrm{SP})^{\beta_0}_{\gamma_{\max}}$ has a solution (x_0, u_0) in $W^{1,\infty}_n([0,1]) \times L^\infty_m([0,1])$ and there are associated Lagrange multipliers verifying (14)–(17). By virtue of (14.2), u_0 is smooth (no commutation) and defined on $[0, 1]$ by (13). Let then $\tilde{H}$ be the augmented hamiltonian

$$\tilde{H}(x, u, p, \mu, \beta) = \beta(p|f_0(x) + B(x)u) + 1/2\mu(|u|^2 - \gamma^2_{\max})$$

where μ is the scalar multiplier associated with the constraint on the control

$$C(u) \leq 0 \tag{22}$$

taking $C(u)=1/2(|u|^2-\gamma_{\max}^2)$. One has $\nabla_u\tilde{H}(x_0,u_0,p_0,\mu_0,\beta_0)=0$ with

$$\mu_0 = \beta/\gamma_{\max}|{}^tB(x_0)p_0| \geq 0$$

The Lagrange multipliers are thus regular enough: p_0 belongs to $W_n^{1,\infty}([0,1])$ and μ_0 belongs to $L^\infty([0,1])$. Furthermore, μ_0 is smooth and (I4.2) implies $\mu_0(t)>0$ on $[0,1]$, so that strict complementarity holds. Besides, since

$$\nabla_{uu}^2\tilde{H}(x_0,u_0,p_0,\mu_0,\beta_0) = \mu_0\mathrm{I}_2 > 0$$

(I_2 denoting the identity matrix of order 2), the strict Legendre–Clebsch condition is fulfilled. Finally, with (I4.3) and (I4.4), all the assumptions of [10] are true and β_0 has a open neighbourhood $V \subset]0,\bar{\beta}[$ on which the mappings

$$\beta \mapsto (x(\cdot,\beta),u(\cdot,\beta)) \in W_n^{1,\infty}([0,1]) \times L_m^\infty([0,1])$$
$$\beta \mapsto (p(\cdot,\beta),\nu(\beta),\mu(\cdot,\beta)) \in W_n^{1,\infty}([0,1]) \times \mathbf{R}^n \times L^\infty([0,1])$$

are continuously differentiable. Then, Lemma 1 applies to $(\mathrm{SP})_{\gamma_{\max}}^\beta$ on V with $z=(x,u)$, $\lambda=(-p,\nu,\mu)$, $\mathcal{Z}=\mathcal{X}\times\mathcal{U}$, $\mathcal{X}=W_n^{1,\infty}([0,1])$, $\mathcal{U}=L_m^\infty([0,1])$, $\mathcal{Y}=L_n^\infty([0,1])\times\mathbf{R}^n\times L^\infty([0,1])$, and

$$J(z,\beta) = \frac{1}{2}|h(x(1))|^2$$
$$F(z,\beta) = \begin{bmatrix} \dot{x}-\beta f(x,u) \\ x(0)-x^0 \\ C(u) \end{bmatrix}$$

Indeed, the constraint (22) is always (strongly) active, and λ does belong to $\mathcal{Y}'$ since, $\mathcal{Y}$ being densely and continuously imbedded into the Hilbert space $\hat{\mathcal{Y}}=L_n^2([0,1])\times\mathbf{R}^n\times L^2([0,1])$, one has $\mathcal{Y}\subset\hat{\mathcal{Y}}\subset\mathcal{Y}'$ (with the usual identification $\hat{\mathcal{Y}}'\simeq\hat{\mathcal{Y}}$). Hence, for $\beta\in V$,

$$\begin{aligned}
\phi'(\beta) &= \lambda(\beta)\partial_\beta F(z(\beta),\beta) \\
&= \langle p(\cdot,\beta), f(x(\cdot,\beta),u(\cdot,\beta))\rangle_{(L^\infty)',L^\infty} \\
&= (p(\cdot,\beta)|f(x(\cdot,\beta),u(\cdot,\beta)))_{L^2} \\
&= \int_0^1 (p(t,\beta)|f(x(t,\beta),u(t,\beta)))dt \\
&= H(0,\beta)/\beta
\end{aligned}$$

thanks to the additional regularity of multipliers (and the constancy of the hamiltonian because the problem is autonomous). The result is true in the neighbourhood of an arbitrary point in $]0, \bar{\beta}[$, and thus holds on the whole open interval. ∎

5. NUMERICAL RESULTS

The numerical resolution of the transfer problem is done in three steps. At the top level, a continuation procedure on the maximum acceleration $\gamma_{\max}$ is employed. So as to reach very low thrusts, we start from strong ones and use the results to initiate the resolution for lower ones: if $\gamma_{\max}^c$ defines the current constraint on the command, if $\bar{\beta}^c$ is the associated solution, we use $\bar{\beta}^c$ to initialize the nonlinear search process for $(\mathrm{SP})_{\gamma_{\max}^+}$ where $\gamma_{\max}^+$ is the next constraint. Continuity properties of the mapping thus defined

$$\gamma \rightarrow \bar{\beta}(\gamma)$$

as well as convergence of the associated optimal states and controls, are studied in [5]. The continuation is stopped when the desired thrust is reached. The choice of the intermediary thrusts is heuristic (see Tab. I). At the second level, for each thrust we take advantage of the fact that $\bar{\beta}^c$ is smaller than $\bar{\beta}^+$ to use the technique described in Sections 3 and 4: the first zero of ϕ is sought by solving the scalar

TABLE I Maximum thrusts and resulting minimum transfer times

F_{max} *(Newtons)*	t_f *(Hours)*
60	15.205
24	35.939
12	73.278
9	100.84
6	147.00
3	296.97
2	442.11
1.4	630.62
1	887.03
0.7	1340.4
0.5	1767.2
0.3	2960.8
0.2	4426.7

equation $\phi(\beta)=0$. The derivative of ϕ is provided to a Newton algorithm by means of the analytical expression (21). The iterative process is stopped as soon as the current iterate β_k is such that $\phi(\beta_k)\leq\varepsilon$ ($\varepsilon=10^{-8}$ in practice). Finally, at the inner level, the evaluation of ϕ at β_k is done by solving the auxiliary problem $(\mathrm{SP})^{\beta_k}_{\gamma^{\varepsilon}_{\max}}$ with single shooting. The initial adjoint state solution $\bar{p}^0_n$ then initializes the resolution of $(\mathrm{SP})^{\beta_{k+1}}_{\gamma^{\varepsilon}_{\max}}$.

Table I details the sequence of thrusts ($F_{\max}=M\gamma_{\max}$, where M is the mass) used for the continuation procedure from 60 Newtons (strong thrust) down to 0.2 Newton (very low thrust), as well as the resulting optimal transfer times. The satellite is assumed to weigh 1.5 Tons, and the boundary conditions are defined through the following values:

$$
\begin{array}{ll}
P^0 = 11625\,\mathrm{km} & P^f = 42165\,\mathrm{km} \\
e^0_x = 0.75 & e^f_x = 0 \\
e^0_y = 0 & e^f_y = 0 \\
L^0 = \pi\,\mathrm{rad} & M = 1500\,\mathrm{kg} \\
\mu^0 = 398600.47\,\mathrm{km}^3\cdot\mathrm{s}^{-2} &
\end{array}
$$

Figure 2 underlines the experimental statement that the product of the maximum thrust by the resulting optimal time is nearly constant. Figure 3 gives the optimal trajectories and controls obtained with the method for 60, 12, 0.5 and 0.3 Newton.

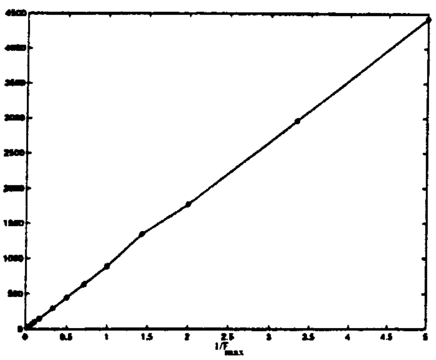

FIGURE 2 Optimal times t_f by inverse of the maximum thrusts $F_{\max}$. The transfer times grow approximately in inverse proportion of the thrusts.

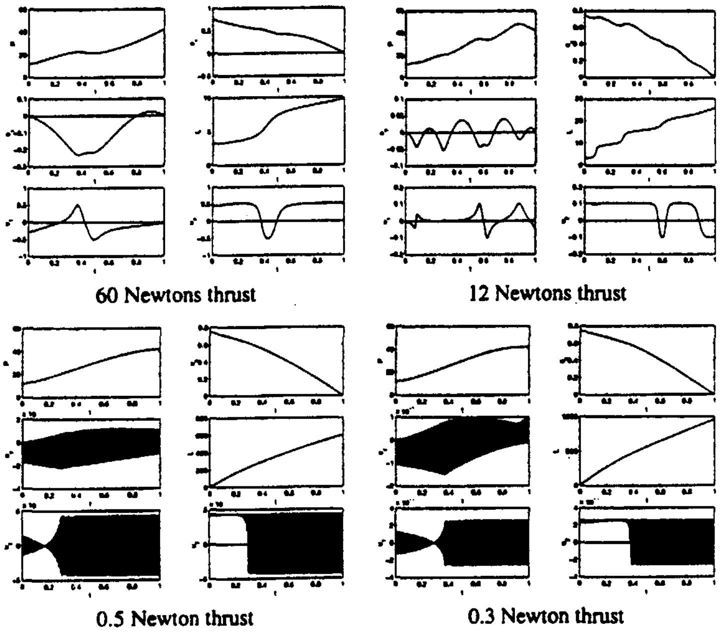

FIGURE 3 Optimal trajectories and controls for several thrusts: $x=(P,e_x,e_y,L)$, $u=(u_1,u_2)$. We notice two distinct phases: first the semi-latus rectum P is increased, then the eccentricity $e=(e_x,e_y)$ is corrected.

The assumptions of Section 4 are also verified numerically: (I4.3) when solving the shooting equation (inversibility of the jacobian at the solution), (I4.4) by integrating backwards the system

$$\dot{y} = \xi(y(t),\beta) \tag{23}$$

$$\dot{Q} = -QA(t,\beta) - {}^{t}A(t,\beta)Q + QB(t,\beta)Q - C(t,\beta) \tag{24}$$

$$y(1) = (x(1,\beta),p(1,\beta)) \tag{25}$$

$$Q(1) = 0_4 \tag{26}$$

once determined the solution $(x(\cdot,\beta),p(\cdot,\beta))$ of $(\mathrm{BVP})_\beta$. Hereabove, 0_4 is the fourth order zero matrix (this choice ensures the positive definiteness in (20)). The second member of the Riccati equation contained in (23)–(26) which requires the computation of the second order derivatives of the dynamics (1)–(4) have been evaluated by automatic differentiation thanks to the *Adifor* code [1]. (I4.3) turns out to be always fulfilled, and Figure 4 gives the evaluations of ϕ and ϕ' (thanks to (21)) as well as the validity domains of (I4.4) for various thrusts. At last, Figure 5 shows that assumption (I4.2) is numerically verified, though for low thrusts there happens to be one point which is almost a commutation.

In so far as ϕ is evaluated using the first order condition (single shooting on the auxiliary problems), it may happen that only an upper estimation of the function is obtained (related to a local minimum of $(\mathrm{SP})^{\beta}_{\gamma_{\max}}$). The inequality (9) then makes it possible to detect such a case (see Fig. 6). Nonetheless, even if for this reason we cannot assert that $\bar{\beta}$ is the absolute minimum of $(\mathrm{SP})_{\gamma_{\max}}$ (it may just be the first root of an upper estimation of ϕ), the method provides an ordered search of

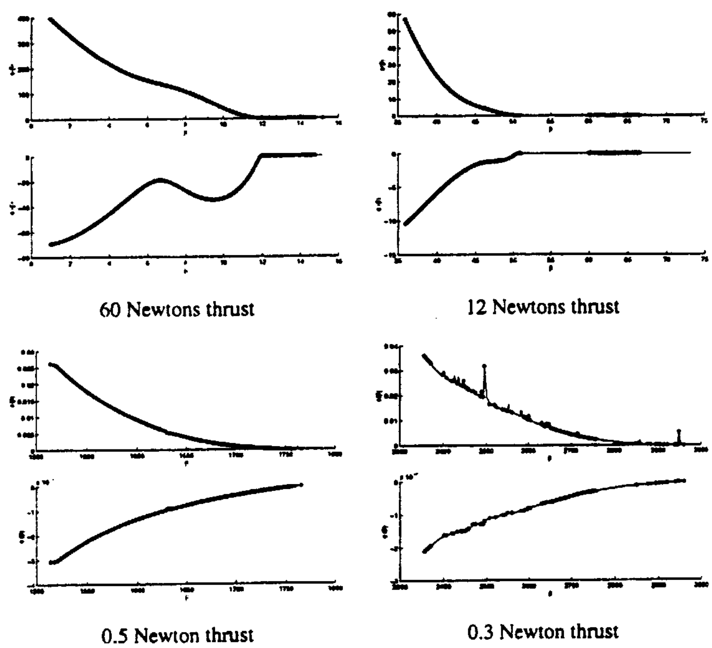

FIGURE 4 Evaluations of ϕ and ϕ' between β_0 (initialization of the Newton search) and $\bar{\beta}$ (solution). The points where the coercivity condition has been checked (that is points where the Riccati equation (23)–(26) has been successfully integrated) are marked with symbol $*$.

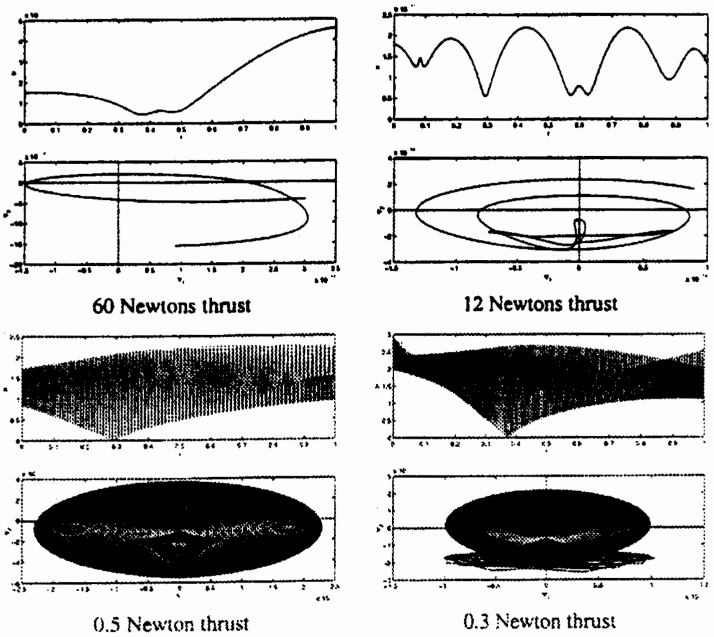

FIGURE 5 The Lagrange multiplier μ associated with the constraint on the modulus of u and the 'commutation function' ψ (which passes through the origin if and only if there is a commutation) are represented. For low thrusts, there is one point where μ and ψ are very close to zero; This point, like all the rapid variations on the control (see Fig. 3), is precisely situated at the perigee and corresponds to the change of phase in the command.

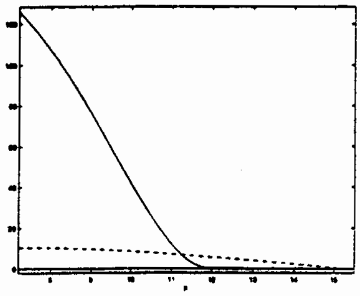

FIGURE 6 The non-controllability function ϕ for 60 Newtons is in solid line, the function mapping β to $1/2|h(\bar{x}(\beta))|^2$ in dashed line. For $\beta < \bar{\beta} \simeq 11.34$ ($\bar{\beta} \simeq 15.2$), the computation only provides an upper bound of ϕ.

the optimal index performance, *thus avoiding too coarse local minima.* Over and above, in addition to this theoretical virtue, the approach has two numerical advantages. First, the domain of convergence is larger than the single shooting one: the approximation provided by the continuation on $\gamma_{\max}$ permits to initialize the iterative process. On the opposite, when using mere shooting (single or multiple), it becomes compulsary to use the heuristic $\bar{t}_f F_{\max} \simeq \text{cst}$ to initialize very precisely the resolution (which in turn may lead to local minima [3]). Besides, though the evaluation of ϕ is done by single shooting on $(\text{SP})^{\beta}_{\gamma_{\max}}$, we break off significatively from the well-known sensitivity of this technique to the initialization of the adjoint initial state. Indeed, during the first iterations of the resolution of $\phi(\beta) = 0$, β is far from $\bar{\beta}$, and no precise approximation of p^0 is available: the auxiliary problems are thus unaccurately solved by shooting, and the evaluation of ϕ is distorted. However, this approximation appears to be *sufficient* to initialize the iterative procedure. Furthermore, as the iterates get closer to $\bar{\beta}$, the shooting problems are solved more and more accurately: the

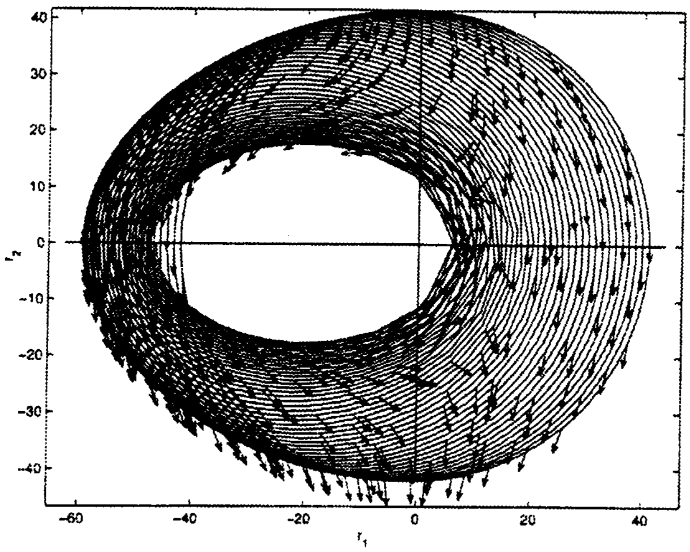

FIGURE 7 Trajectory for $F_{\max} = 1$ Newton in the geocentric frame. The arrows represent the action of the optimal control. Starting from a very eccentric low orbit with high velocity near the perigee, the semilatus is increased, and then the eccentricity is corrected for the insertion on the circular high orbit.

computation of ϕ becomes more precise, and finally leads to the solution. Hence, paid the price of an additional iteration level (and then of an increased amount of computation), an effective gain in robustness over classical methods is obtained.

6. CONCLUSION

We have proposed a new method, based on a so-called non-controllability function. Thanks to a sensitivity analysis of the auxiliary parametric optimal control problems associated with the method, regularity properties of the function are proved, and the technique is successfully applied to the minimum time transfer problem (see Fig. 7): optimal trajectories are obtain for very low thrusts, even below 0.3 Newton. Besides, the method has proved to be more robust than classical indirect approaches, being less sensitive to the initialization of the criterion and of the initial adjoint state. The extension of the approach to other performance indexes (*e.g.*, maximization of the mass of the satellite) and to more realistic models taking into account the variation of the mass is currently worked out.

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