

Averaging and optimal control of elliptic Keplerian orbits with low propulsion[☆]

Bernard Bonnard^{a,*}, Jean-Baptiste Caillau^b, Romain Dujol^b

^a*Institut de Mathématiques de Bourgogne (UMR CNRS 5584), Dijon, France*

^b*ENSEEIH-IRIT (UMR CNRS 5505), Toulouse, France*

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Abstract

This article deals with the optimal transfer of a satellite between Keplerian orbits using low propulsion. It is based on preliminary results of Geffroy [Généralisation des techniques de moyennation en contrôle optimal, application aux problèmes de rendez-vous orbitaux à poussée faible, Ph.D. Thesis, Institut National Polytechnique de Toulouse, France, Octobre 1997] where the optimal trajectories are approximated using averaging techniques. The objective is to introduce the appropriate geometric framework and to complete the analysis of the averaged optimal trajectories for energy minimization, showing in particular the connection with Riemannian problems having integrable geodesics.

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1. Introduction

An important problem in astronautics is to transfer a satellite between elliptic orbits. A modern challenge concerns orbit transfers with electro-ionic propulsion and very low thrust. For the sake of simplicity, we restrict ourselves to coplanar transfers, assuming moreover the mass constant. If we decompose the thrust in the *tangential-normal frame*, the system is described by Gauss equations:

$$\dot{e} = \frac{\sqrt{1-e^2}}{\sqrt[3]{\mu n}} \frac{1}{\sqrt{1+2e\cos v+e^2}} \times \left[2(e+\cos v)u_t - \sin v \frac{1-e^2}{1+e\cos v} u_n \right], \quad (1)$$

$$\dot{n} = -\frac{3n^{2/3}}{\sqrt{1-e^2}\sqrt[3]{\mu}} \left[\sqrt{1+2e\cos v+e^2} u_t \right], \quad (2)$$

$$\dot{\omega} = \frac{\sqrt{1-e^2}}{e^3\sqrt[3]{\mu n}} \frac{1}{\sqrt{1+2e\cos v+e^2}} \times \left[(2\sin v)u_t + \frac{2e+\cos v+e^2\cos v}{1+e\cos v} u_n \right], \quad (3)$$

$$\dot{l} = (1+e\cos v)^2 \frac{n}{(1-e^2)^{3/2}}, \quad (4)$$

where the coordinates are e , the *eccentricity*, n , the *mean motion* ($n = \sqrt{\mu/a^3}$, a semi-major axis and μ gravitation constant), ω , the *argument of the pericenter*, l , the polar angle or *longitude*, and $v = l - \omega$ the *true anomaly*. The control $u = (u_t, u_n)$ is decomposed in the *tangential-normal frame*. We observe that if the thrust $|u| \leq \varepsilon$ is low, the system can be renormalized according to

$$\dot{x} = \varepsilon \sum_{i=1}^2 u_i F_i(l, x), \quad (5)$$

$$\dot{l} = G(l, x), \quad (6)$$

where $x = (e, n, \omega)$ are first integrals of the uncontrolled motion that is *slow variables*, l being the *fast variable* and $u = (u_1, u_2)$ the control vector, $|u| \leq 1$. The functions F_i, G are 2π -periodic with respect to the angular variable.

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* Corresponding author. Tel.: +33 3 80 39 58 78; fax: +33 3 80 39 58 99.
E-mail address: bernard.bonnard@u-bourgogne.fr (B. Bonnard).

In orbital transfer, we must steer the system from an initial position (x_0, l_0) to a terminal orbit represented by x_1 , taking into account physical cost functions, e.g. the time, the energy $\int_0^T \sum u_i^2 dt$, or the final mass related to $\int_0^T |u| dt$. Using the maximum principle, the optimal trajectories are to be found among a set of extremal curves, solution of a Hamiltonian system defined by a Hamiltonian H_ε . Such a system is extremely complicated and can be analyzed only using numerical simulations. Moreover, for low propulsion we essentially observe on the numerical results the averaged behavior of the solutions. This was the starting point of Geffroy's work which provides a preliminary analysis of the averaged system for the energy minimization problem (the constraint $|u| \leq 1$ being relaxed but satisfied in the end by adjusting the transfer time). This led to an averaged system which can be mathematically computed using standard integral evaluations and is integrable by quadrature if we transfer the system towards a geostationary orbit. The aim of this short article is to complete the computations and to derive properties of the optimal solution. Moreover, we show that the averaged system is equivalent to a Riemannian problem in \mathbf{R}^3 which can be written

$$\dot{x} = \sum_{i=1}^3 u_i F_i(x), \quad \int_0^T \sum_{i=1}^3 u_i^2 dt \rightarrow \min.$$

The additional control being generated by averaging the optimal control with respect to the fast variable and producing displacement in the directions generated by the Lie brackets.

The merits of this study are twofold. First of all, from [1, Theorem, p. 294, Chapter 10] the averaged extremal curves (respectively the extremal cost) will provide an approximation of order ε (respectively ε^2 , see [5]) of the extremals of the energy minimization and the corresponding cost. They can thus be used as a starting guess for a continuation method applied to time minimization or mass maximization. Besides, the averaged system provides a neat geometric insight into the orbit transfer problems in general.

2. Averaging as a Riemannian approximation of the problem

2.1. Averaged system

Setting $u_t = \varepsilon v_t$, $u_n = \varepsilon v_n$, where $|u| \leq \varepsilon$ and parameterizing the trajectories by the *cumulated longitude* l , the energy minimization problem takes the form:

$$\frac{de}{dl} = \varepsilon D \left[2(e + \cos v)v_t - \sin v \frac{1 - e^2}{W} v_n \right], \quad (7)$$

$$\frac{dn}{dl} = -3\varepsilon \frac{1 - e^2}{\sqrt[3]{\mu}} \left[\frac{\sqrt{1 + 2e \cos v + e^2}}{W^2} v_t \right], \quad (8)$$

$$\frac{d\omega}{dl} = \varepsilon \frac{D}{e} \left[(2 \sin v)v_t + \frac{2e + \cos v + e^2 \cos v}{W} v_n \right]. \quad (9)$$

The cost is

$$\int_{l_0}^{l_1} \varepsilon^2 (v_t^2 + v_n^2) \frac{(1 - e^2)^{3/2}}{nW^2} dl,$$

where

$$W = 1 + e \cos v, \quad D = \frac{(1 - e^2)^2}{n^{4/3} W^2 \sqrt{1 + 2e \cos v + e^2}}$$

and the control has to satisfy the constraint $v_t^2 + v_n^2 \leq 1$. The problem can be formulated as

$$\frac{dx}{dl} = \sum_{i=1}^2 u_i F_i(l, x), \quad \min_{u_1^2 + u_2^2 \leq 1} \int_{l_0}^{l_1} \varepsilon^2 (u_1^2 + u_2^2) g(l, x) dt$$

and from the maximum principle, the associated Hamiltonian is

$$H_\varepsilon = p^0 \varepsilon^2 (u_1^2 + u_2^2) g(l, x) + \varepsilon \sum_{i=1}^2 u_i \langle p, F_i(l, x) \rangle,$$

where p^0 can be normalized to $-1/\varepsilon < 0$ (*normal case*) and p is the adjoint vector dual to x . Relaxing the constraint $u_1^2 + u_2^2 \leq 1$, leads to compute the extremals solving $\partial H_\varepsilon / \partial u = 0$.

Plugging the corresponding solutions in H_ε defines the *true Hamiltonian* function

$$\begin{aligned} H_\varepsilon(l, x, p) &= \frac{\varepsilon n W^2}{4(1 - e^2)^{3/2}} \left[\left(2p_1 D(e + \cos v) \right. \right. \\ &\quad \left. \left. - 3p_2 \frac{1 - e^2}{\sqrt[3]{n}} \frac{\sqrt{1 + 2e \cos v + e^2}}{W^2} + 2p_3 \frac{D \sin v}{e} \right)^2 \right. \\ &\quad \left. + \left(-p_1 D \sin v \frac{1 - e^2}{W} \right. \right. \\ &\quad \left. \left. + p_3 \frac{D(2e + \cos v + e^2 \cos v)}{eW} \right)^2 \right]. \quad (10) \end{aligned}$$

By construction, H_ε is 2π -periodic with respect to the angular variable l , and the mean Hamiltonian is

$$\bar{H}(x, p) = \frac{1}{2\pi\varepsilon} \int_0^{2\pi} H_\varepsilon(l, x, p) dl$$

and the averaged can be computed with respect to the true anomaly $v = l - \omega$.

A key observation is that the averaging process amounts to evaluating integrals $\int_0^{2\pi} F(\cos v, \sin v)$ where F is a rational function. Therefore, a standard residue computation gives the following result.

Proposition 1. *The averaged Hamiltonian is*

$$\bar{H} = \frac{1}{8N^{5/3}} \left[5(1 - E^2)P_1^2 + 18N^2P_2^2 + \frac{5 - 4E^2}{E^2}P_3^2 \right], \quad (11)$$

where (E, N, Ω) is the notation for the variables (e, n, ω) of the averaged system and (P_1, P_2, P_3) are the corresponding dual variables.

2.2. Associated Riemannian problem

We observe that \bar{H} can be written as the sum of three squares, $\bar{H} = (1/2)(H_1^2 + H_2^2 + H_3^2)$, where H_1, H_2 and H_3 are the Hamiltonian lifts $H_i = \langle p, F_i \rangle$ associated to the following vector fields:

$$F_1 = \sqrt{\frac{5}{4}(1 - E^2)} \frac{1}{N^{5/6}} \frac{\partial}{\partial E}, \quad (12)$$

$$F_2 = \sqrt{\frac{9}{2}} N^{1/6} \frac{\partial}{\partial N}, \quad (13)$$

$$F_3 = \frac{1}{2} \frac{\sqrt{5 - 4E^2}}{E} \frac{\partial}{\partial \Omega}, \quad (14)$$

where $N > 0$ and $E > 0$, the singularity $E = 0$ corresponding to a circular orbit. Therefore, \bar{H} is the Hamiltonian associated to the Riemannian problem

$$\dot{x} = \sum_{i=1}^3 u_i F_i(x), \quad x = (E, N, \Omega), \quad \int_0^T \sqrt{\sum_{i=1}^3 u_i^2} dt \rightarrow \min \quad (15)$$

and according to Maupertuis principle, the length minimization is equivalent to the energy minimization

$$\int_0^T \sum_{i=1}^3 u_i^2 dt \rightarrow \min, \quad (16)$$

where $T > 0$ is fixed.

The existence of such—in general—sub-Riemannian equivalent problems, holds for the whole class of systems normalizable to

$$\frac{dx}{dl} = \varepsilon \sum_{i=1}^m u_i F_i(l, x), \quad x \in \mathbf{R}^n, \quad \varepsilon^2 \int_0^T \sum_{i=1}^m u_i^2 dt \rightarrow \min, \quad (17)$$

where the Hamiltonian is $H_\varepsilon = \frac{1}{2} \sum_{i=1}^m \langle p, F_i(l, x) \rangle^2$ with $H_\varepsilon \geq 0$ and hence $\bar{H} \geq 0$. If we can write $\bar{H} = \frac{1}{2} \sum_{i=1}^n H_i^2$, n being the dimension of the space, $H_i = \langle p, F_i \rangle$, the averaged is Riemannian. This is also the case for the non-coplanar orbit transfer problem, see [2].

3. Integrability of the averaged system

3.1. Geometric preliminaries

The equations associated to \bar{H} are:

$$\frac{dE}{dt} = \frac{\partial \bar{H}}{\partial P_1} = \frac{5}{4} P_1 \frac{1 - E^2}{N^{5/3}}, \quad (18)$$

$$\frac{dN}{dt} = \frac{\partial \bar{H}}{\partial P_2} = \frac{9}{2} P_2 N^{1/3}, \quad (19)$$

$$\frac{d\Omega}{dt} = \frac{\partial \bar{H}}{\partial P_3} = \frac{1}{4} P_3 \frac{5 - 4E^2}{E^2} \frac{1}{N^{5/3}}, \quad (20)$$

$$\frac{dP_1}{dt} = -\frac{\partial \bar{H}}{\partial E} = \frac{5}{4} \left(P_1^2 E + \frac{P_3^2}{E^3} \right) \frac{1}{N^{5/3}}, \quad (21)$$

$$\begin{aligned} \frac{dP_2}{dt} = -\frac{\partial \bar{H}}{\partial N} = & \frac{25}{24} P_1^2 \frac{1 - E^2}{N^{8/3}}, -\frac{3}{4} \frac{P_2^2}{N^{2/3}} \\ & + \frac{5}{24} P_3^2 \frac{5 - 4E^2}{E^2} \frac{1}{N^{8/3}}, \end{aligned} \quad (22)$$

$$\frac{dP_3}{dt} = -\frac{\partial \bar{H}}{\partial \Omega} = 0. \quad (23)$$

They describe the extremal curves of the Riemannian metric

$$\bar{g} = \frac{2}{9N^{1/3}} dN^2 + \frac{4}{5} \frac{N^{5/3}}{1 - E^2} dE^2 + 4 \frac{N^{5/3}}{5 - 4E^2} E^2 d\Omega^2.$$

In particular, the orbit elements N, E and Ω are *orthogonal* coordinates. The metric in the previous orthogonal coordinates is singular for $E = 0$, which corresponds to circular orbit. This singularity arises from a polar transformation centered on circular orbits and can be removed using equinoctial variables. We observe that since the Hamiltonian does not depend on Ω , the variable is cyclic and its dual P_3 is a first integral of the averaged motion.

Geometrically, the condition $P_3 = 0$ is the transversality condition of an optimal transfer towards a circular orbit where the angle Ω of the pericenter is not specified. This is the case for the important problem of steering the system to a *geostationary orbit*. In this occurrence, we can restrict the system to the invariant symplectic space $\{E, N, P_1, P_2\}$ and the reduced Hamiltonian is related to a planar Riemannian metric.

Next, we give normalized coordinates which are crucial to analyze the solutions and decide about optimality.

3.2. Normal coordinates

Proposition 2. *The metric \bar{g} is isometric to $dv^2 + v^2(dw^2 + G(w) d\Omega^2)$.*

Proof. The main step is to normalize the two-dimensional metric

$$g = \frac{2}{9N^{1/3}} dN^2 + \frac{4}{5} \frac{N^{5/3}}{1 - E^2} dE^2$$

associated to transfer towards circular orbits. We set

$$v = \frac{2\sqrt{2}}{5}N^{5/6} \quad \text{and} \quad w = \sqrt{\frac{5}{2}} \arcsin E$$

and g takes the so-called *polar form* $g = dv^2 + v^2 dw^2$.

Hence $E = \sin(\sqrt{2/5}w)$ and a straightforward computation gives us

$$G(w) = \frac{25}{2} \frac{\sin^2(\sqrt{2/5}w)}{1 + 4 \cos^2(\sqrt{2/5}w)}.$$

The transformation is well-defined for $N > 0$ and $w \in [0, \pi\sqrt{5}/(2\sqrt{2})]$. \square

3.3. Averaged transfer to circular orbits

In this case the metric is reduced to the polar form $g = dv^2 + v^2 dw^2$. If we set $x = v \cos w$, $y = v \sin w$, it is isometric to $dx^2 + dy^2$. We get the following result.

Proposition 3. *In suitable coordinates, optimal curves corresponding to transfer to circular orbits are straight lines.*

In particular, the geodesics curves are integrable and this result was already obtained in [6], using symbolic computation. Another consequence of our normal form is to extend this integrability property to the full system. This is presented hereafter using the orbit elements.

3.4. Integrability and parameterization

Since Ω is a cyclic coordinate, $P_3 = C_1$, constant. Introducing $U = NP_2$, we deduce from the equations that

$$\dot{U} = \frac{1}{N^{5/3}} \left[\frac{25}{24}(1 - E^2)P_1^2 + \frac{15}{4}N^2P_2^2 + \frac{5}{24} \frac{5 - 4E^2}{E^2} P_3^2 \right].$$

Since $\bar{H} = C_2$, constant, we have

$$8C_2 = \frac{1}{N^{5/3}} \left[5(1 - E^2)P_1^2 + 18N^2P_2^2 + \frac{5 - 4E^2}{E^2} P_3^2 \right],$$

which implies $\dot{U} = \frac{5}{3}C_2$. If we set $V = N^{5/3}$, we obtain

$$\dot{V} = \frac{5}{3}N^{2/3}\dot{N} = \frac{15}{2}U,$$

therefore, $\dot{V} = \frac{25}{2}C_2$. Hence, $V(t)$ is a second-order polynomial given by

$$V(t) = \frac{25}{4}C_2t^2 + \dot{V}(0)t + V(0)$$

and, with (19), this gives N and P_2 . In order to compute the remaining, we introduce

$$\bar{H}' = 5(1 - E^2)P_1^2 + \frac{5 - 4E^2}{E^2} P_3^2.$$

The corresponding Hamiltonian system in the symplectic space $\{E, \Omega, P_1, P_3\}$ is integrated using the parameterization $dT = dt/(8N^{5/3})$. In normal coordinates, this amounts to computing

the geodesics of the Riemannian metric $dw^2 + G(w) d\Omega^2$. Such a geodesic flow is integrable because it is a Liouville metric. Equivalently, the existence of quadratures comes from the fact that the associated Hamiltonian system \bar{H}' admits two independent and commuting first integrals: \bar{H}' itself, and the dual variable P_3 .

To integrate, we proceed as follows. The associated extremals equations are:

$$\frac{dE}{dT} = 10(1 - E^2)P_1,$$

$$\frac{d\Omega}{dT} = 2 \frac{5 - 4E^2}{E^2} P_3,$$

$$\frac{dP_1}{dT} = 10 \left(EP_1^2 + \frac{P_3^2}{E^3} \right),$$

$$\frac{dP_3}{dT} = 0.$$

Since Ω is cyclic, $P_3 = C_1$, and \bar{H}' is a Hamiltonian function of the two symplectic variables (E, P_1) depending upon the parameter C_1 . The associated planar system is completely integrable and Ω can be computed by quadrature. In order to obtain a parameterization, we set $\bar{H}' = C_3^2$ and use $P_1 = (dE/dT)/[10(1 - E^2)]$. We get

$$\left(\frac{dE}{dT} \right)^2 = \frac{20(1 - E^2)}{E^2} [C_3^2 E^2 - (5 - 4E^2)C_1^2].$$

Introducing $W = 1 - E^2$,

$$\left(\frac{dW}{dT} \right)^2 = Q(W),$$

where Q is the polynomial of degree 2, $Q(W) = 80W[(C_3^2 - C_1^2) - (C_3^2 + 4C_1^2)W]$ whose discriminant is strictly positive. Hence, the solution is (with φ determined by the initial conditions):

$$W = \frac{1}{2} \frac{C_3^2 - C_1^2}{C_3^2 + 4C_1^2} [1 + \sin(4\sqrt{5}(C_3^2 + 4C_1^2)^{1/2}T + \varphi)].$$

We deduce:

$$\Omega(T) = \Omega(0) + 2C_1 \int_0^T \frac{1 + 4W(s)}{1 - W(s)} ds.$$

Proposition 4. *If $V = N^{5/3}$ and $W = 1 - E^2$, the extremal solutions are parameterized by*

$$V(t) = (25/4)C_2t^2 + \dot{V}(0)t + V(0),$$

where $\bar{H} = C_2$, $\dot{V}(0) = \frac{15}{2}N(0)P_2(0)$,

$$W = \frac{1}{2} \frac{C_3^2 - C_1^2}{C_3^2 + 4C_1^2} t [1 + \sin(4\sqrt{5}(C_3^2 + 4C_1^2)^{1/2}T + \varphi)],$$

where $\bar{H}' = C_3^2$ and $P_3 = C_1$,

$$\Omega(T) = \Omega(0) + 2C_1 \int_0^T \frac{1 + 4W(s)}{1 - W(s)} ds$$

and $T = \int_0^T ds/(8V(s))$.

Hence, in conclusion of this section, we get the following result:

Theorem 5. *The averaged system corresponding to a coplanar transfer is completely integrable by quadratures. The value function, solution of Hamilton–Jacobi equation, can be computed and corresponds to the energy function of the associated Riemannian problem.*

4. Riemannian spheres of the averaged coplanar transfer

4.1. Preliminaries

Let $x = (N, E, \Omega)$ and $p = (P_1, P_2, P_3)$. We denote by

$$z(t, z_0) = (x(t, x_0, p_0), p(t, x_0, p_0))$$

the extremal curve corresponding to \bar{H} , solution of Eqs. (18)–(23) with initial condition $z_0 = (x_0, p_0)$. We parameterize curves by arc-length on the fixed level set $\bar{H} = \frac{1}{2}$. Given x_0 , the exponential mapping is the map $\exp_{x_0} : (p_0, t) \mapsto x(t, x_0, p_0)$. Let (p_0, t_c) , $t_c > 0$, be a point where the exponential mapping is not an immersion. Then t_c is called a conjugate time and the image is called a conjugate point. The conjugate locus $C(x_0)$ is the set of first conjugate points. For a given extremal curve, the cut point is the first point where the curve ceases to be optimal. When we consider all the extremals starting from x_0 , the set of such points forms the cut locus $L(x_0)$.

The Riemannian sphere with radius $r > 0$ is the set $S(x_0, r)$ of points which are at (Riemannian) distance r from x_0 , and the Riemannian ball is the set $B(x_0, r)$ of points at distance less or equal to r from x_0 . If $r > 0$ is small enough, the sphere is formed by extremities of extremal curves. The Riemannian sphere is sub-analytic if the metric is analytic and the singularities correspond to cut points which can be either conjugate points or points where two minimizing extremal curves with same length intersect. In the Riemannian case, the sphere is always smooth for r small enough, contrarily to the sub-Riemannian case where conjugate points accumulate at x_0 [4].

Our program is to investigate the spheres of increasing radius, related to coplanar transfer, integrating the extremal flow, so as to conclude about global optimality. This study is connected to the original energy problem because extremals of the averaged problem are $o(\varepsilon)$ approximations of the extremal curves [1]. We give preliminary results obtained with our geometric analysis and numerical computations. In particular, the conjugate locus can be computed using the `cotcot` algorithm presented in [3]. In the numerical computation, we must take into account the singularity corresponding to circular orbits at $E = 0$. This singularity can be removed by introducing the eccentricity vector $(E \cos \Omega, E \sin \Omega)$ and the curves can be prolonged beyond circular orbits. On the opposite, the singularity $E = \pm 1$ cannot be removed: it corresponds to points where extremals leave the elliptic domain, $E = \pm 1$ being associated to parabolic trajectories of Kepler equation.

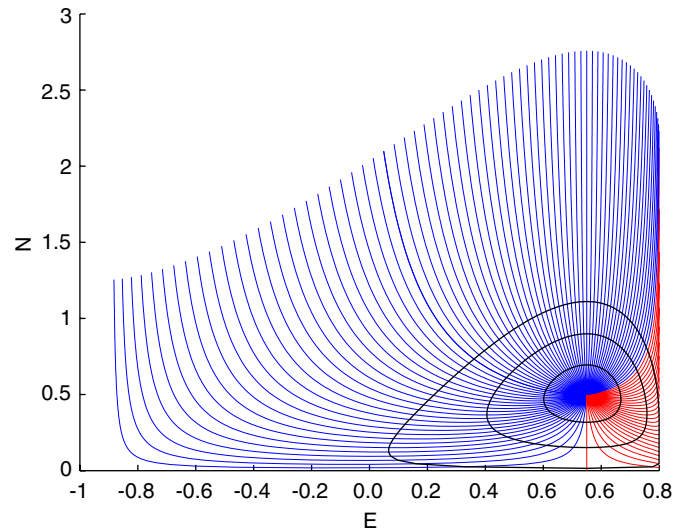


Fig. 1. Extremal curves towards circular orbits. In coordinates (E, N, Ω) , the initial point is $(7.5e - 1, 5e - 1, 0)$. Spheres are presented for radii $1e - 1$, $2e - 1$ and $3e - 1$.

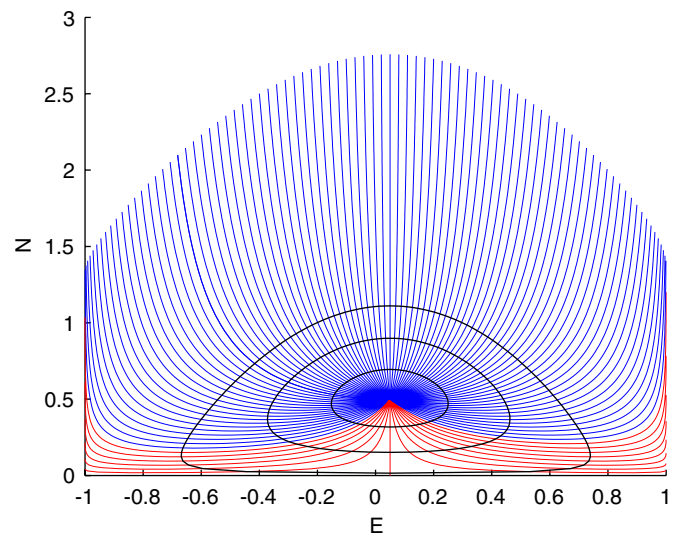


Fig. 2. Extremal curves towards circular orbits. In coordinates (E, N, Ω) , the initial point is $(5e - 2, 5e - 1, 0)$. Spheres are presented for radii $1e - 1$, $2e - 1$ and $3e - 1$.

4.2. Coplanar transfer towards circular orbits

In Figs. 1 and 2, we represent extremal curves restricted to (E, N) and corresponding to transfer to circular orbits. The associated extremals are straight lines in suitable coordinates in order that they are globally optimal. The line $E = 0$ corresponds to circular orbits with unprescribed argument of the pericenter Ω . The extremals can be prolonged and E becomes negative. Geometrically this corresponds to a π -singularity where the argument of the pericenter rotates of π , which can be seen using the eccentricity vector $(E \cos \Omega, E \sin \Omega)$. The lines $E = \pm 1$ define the aforementioned singularity where the orbits become elliptic.

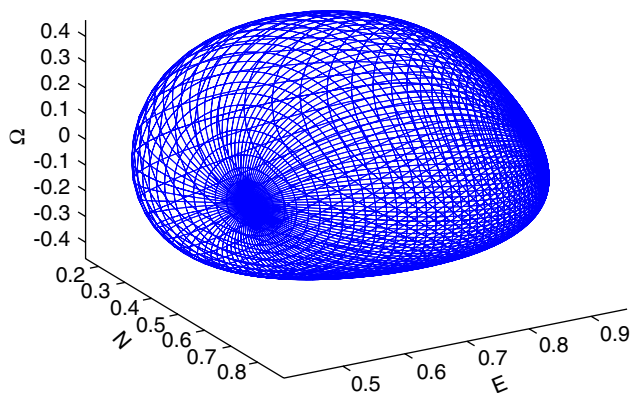


Fig. 3. Sphere of radius $2e - 1$.

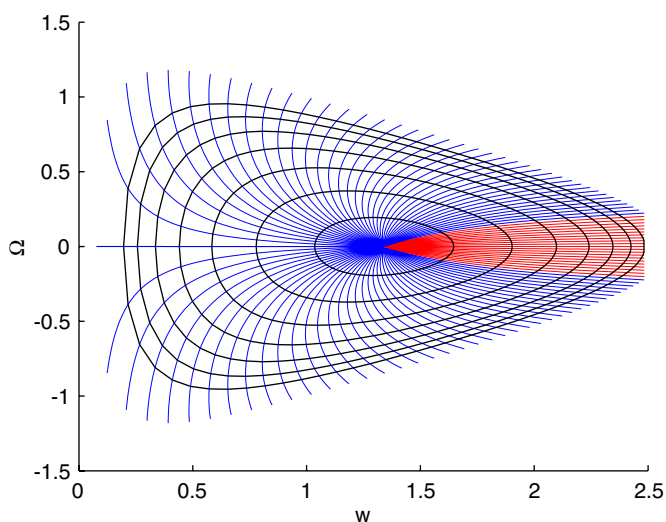


Fig. 4. Extremal curves projected on the (w, Ω) -space up to length 1. The concentric curves are for length $1e - 1, 2e - 1, \dots, 7e - 1$.

4.3. The general case

Having solved the transfer to circular orbits, we can consider the general problem where Ω is not constant. To check optimality, it is sufficient to represent the sphere and to check its smoothness. In Fig. 3 we represent the sphere with center $x_0 = (E(0), N(0), \Omega(0)) = (7.5e - 1, 5e - 1, 0)$ and radius

$2e - 1$. The global analysis can be deduced by projecting on the plane (E, Ω) . In normal coordinates, this amounts to analyzing the extremals of a Riemannian metric $dw^2 + G(w) d\Omega^2$. The function $G(w)$ is related to the Gauss curvature K according to the formula:

$$K = -\frac{1}{\sqrt{G}} \frac{\partial^2 \sqrt{G}}{\partial w^2}.$$

The curvature governs the distribution of conjugate points according to Jacobi equation, and the conjugate locus can be computed. We represent in Fig. 4 the corresponding extremals and observe that the conjugate and cut loci are empty.

5. Conclusion

We have completed in this note preliminary results obtained in [6] for the averaged system associated to the energy minimization problem in the case of coplanar orbit transfer with low propulsion. Our contribution is twofold. First, we connect the problem to a Riemannian problem. Secondly, we make a geometric analysis of the extremal flow which allows to prove integrability and to make an optimality analysis, using explicit and numerical computations. This analysis provides a first insight into the understanding of the geometry, and in particular the curvature, of the controlled Kepler equation.

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