SIAM J. CONTROL OPTIM.
Vol. 50, No. 6, pp. 3178–3202

MINIMUM TIME CONTROL OF THE RESTRICTED THREE-BODY PROBLEM*

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Abstract. The minimum time control of the circular restricted three-body problem is considered. Controllability is proved on an adequate submanifold. Singularities of the extremal flow are studied by means of a stratification of the switching surface. Properties of homotopy maps in optimal control are framed in a simple case. The analysis is used to perform continuations on the two parameters of the problem: the ratio of masses and the magnitude of the control.

Key words. three-body problem, minimum time control, control-affine systems, homotopy, conjugate points

AMS subject classifications. 49K15, 70Q05

DOI. 10.1137/110847299

1. Problem statement.

The control of the two-body problem was addressed in [16]. The first body exerted a central force on the second, which was an artificial satellite of negligible mass whose thrust was the control. The resulting controlled Kepler equation was shown to be controllable, and minimization of time was studied. (See also the subsequent papers [8, 15].) The present paper is the continuation of this work. Now under the influence of two primary bodies, the artificial satellite is still endowed with a thrust. The motion of the two primaries, not influenced by the third negligible mass, is supposed to be circular. Among the numerous previous studies on space missions in the three-body framework, one has to mention the pioneering work of [20] and more recently the work of [26]. These approaches are purely celestial mechanical ones and rely on a fine knowledge of the dynamical system with three bodies or more. For more on the control side see, e.g., [4] for numerical results using direct methods, [6] for a preliminary study on stabilization, and [28, 29] for a combination of control and dynamical system techniques. We present a purely optimal control approach for time minimization. The indirect methods (shooting) used for numerical computations are driven by the geometric analysis of the problem. The model we consider is the following [35].

Let $\mu \in (0,1)$ be the ratio of the primaries masses, and let $Q_\mu := \mathbf{C}\backslash\{-\mu, 1-\mu\}$. For $q \in Q_\mu \subset \mathbf{C} \simeq \mathbf{R}^2$ and positive thrust magnitude ε, define the controlled circular restricted three-body problem (planar model) according to

$$\ddot{q}(t) - \nabla\Omega_\mu(q(t)) + 2i\dot{q}(t) = \varepsilon u(t), \quad |u(t)| = \sqrt{u_1^2(t) + u_2^2(t)} \le 1.$$

Here, $(q, \dot{q}) \in X_\mu = TQ_\mu \simeq Q_\mu \times \mathbf{C}$ are Cartesian coordinates in a rotating frame

*Received by the editors September 7, 2011; accepted for publication (in revised form) August 13, 2012; published electronically November 15, 2012.
http://www.siam.org/journals/sicon/50-6/84729.html

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($q = e^{-it}\xi$, where ξ is the position vector in a fixed frame) and

$$\Omega_\mu(q) := \frac{1}{2}|q|^2 + \frac{1-\mu}{|q+\mu|} + \frac{\mu}{|q-1+\mu|}\cdot$$

Another choice of coordinates consists in letting $X_\mu = T^*Q_\mu$, taking the cotangent bundle instead to write the uncontrolled part of the dynamics in Hamiltonian form. Let $p = \dot{q} + iq$ and let

$$\begin{aligned} J_\mu(q,\dot{q}) &:= \frac{1}{2}|\dot{q}|^2 - \Omega_\mu(q), \\ &= \frac{1}{2}|p|^2 + p \wedge q - \frac{1-\mu}{|q+\mu|} - \frac{\mu}{|q-1+\mu|} \end{aligned}$$

be the *Jacobian integral*. Then,

$$\dot{q}(t) = \frac{\partial J_\mu}{\partial p}(q(t),p(t)), \quad \dot{p}(t) = -\frac{\partial J_\mu}{\partial q}(q(t),p(t)) + \varepsilon u(t), \quad |u(t)| \le 1.$$

More compactly,

$$\dot{x}(t) = F_0(x(t)) + \varepsilon u_1(t) F_1(x(t)) + \varepsilon u_2(t) F_2(x(t)), \quad |u(t)| \le 1,$$

with, in (q,p) coordinates for $x \in X_\mu$,

$$F_0(q,p) := \overrightarrow{J_\mu}, \quad F_1(q,p) := \frac{\partial}{\partial p_1}, \quad F_2(q,p) := \frac{\partial}{\partial p_2},$$

where the symplectic gradient $\overrightarrow{J_\mu} = (\nabla_p J_\mu, -\nabla_q J_\mu)$ is the *drift* of the system. When $\mu = 0$, we get a two-body problem: $J_0 = E - C$ with energy and momentum, respectively, equal to

$$\begin{aligned} E &:= \frac{1}{2}|\dot{\xi}|^2 - \frac{1}{|\xi|} = \frac{1}{2}|\dot{q}|^2 - \frac{1}{2}|q|^2 - q \wedge \dot{q} - \frac{1}{|q|}, \\ C &:= \xi \wedge \dot{\xi} = q \wedge \dot{q} + |q|^2. \end{aligned}$$

Restricting to the *elliptic domain*, $X_0 \cap \{E < 0,\ C > 0\}$, another system of coordinates tailored for the analysis is obtained using orbital elements describing the geometry of the osculating ellipse. Let $n > 0$ be the *mean motion* ($a^3 n^2 = 1$ if a is the semimajor axis), $(e_x, e_y) \in \mathbf{D}$ be the *eccentricity vector* (where $\mathbf{D}$ is the open unit ball of $\mathbf{R}^2$), and $l \in \mathbf{R}$ be the *longitude* (the class modulo 2π of l is just the polar angle in the fixed (ξ_1, ξ_2)-frame). Alternatively, one can use polar coordinates $(e,\theta) \in (0,1) \times \mathbf{S}^1$ for the eccentricity on the (pointed) Poincaré disk $\mathbf{D}$ (θ is called the *argument of pericenter*). In this system, $x = (n, e, \theta, l)$,

$$\begin{aligned} F_0(x)_{|\mu=0} &= \frac{nW^2}{(1-e^2)^{3/2}}\frac{\partial}{\partial l}, \quad W = 1 + e\cos\tau, \\ \widetilde{F}_1(x) &= \frac{\sqrt{1-e^2}}{n^{1/3}}\left(-\frac{3ne\sin\theta}{1-e^2}\frac{\partial}{\partial n} + \sin\tau\frac{\partial}{\partial e} - \cos\tau\frac{1}{e}\frac{\partial}{\partial\theta}\right), \\ \widetilde{F}_2(x) &= \frac{\sqrt{1-e^2}}{n^{1/3}}\left(-\frac{3nW}{1-e^2}\frac{\partial}{\partial n} + \left(\cos\tau + \frac{e+\cos\tau}{W}\right)\frac{\partial}{\partial e} + \left(\sin\tau + \frac{\sin\tau}{W}\right)\frac{1}{e}\frac{\partial}{\partial\theta}\right), \end{aligned}$$

where $\tau = l - \theta$. We have also used a feedback on the control to express the control not in the $\{\partial/\partial\dot{\xi}_1, \partial/\partial\dot{\xi}_2\}$ frame but in the *radial-orthoradial* one, so

$$\widetilde{F}_1(\xi,\dot{\xi}) = \frac{\xi_1}{|\xi|}\frac{\partial}{\partial\dot{\xi}_1} + \frac{\xi_2}{|\xi|}\frac{\partial}{\partial\dot{\xi}_2}, \quad \widetilde{F}_2(\xi,\dot{\xi}) = -\frac{\xi_2}{|\xi|}\frac{\partial}{\partial\dot{\xi}_1} + \frac{\xi_1}{|\xi|}\frac{\partial}{\partial\dot{\xi}_2}\cdot$$

The criterion under consideration is the final time, and the paper is organized as follows. Section 2 is devoted to controllability. Independently of the bound on the control, it is proved that admissible trajectories between arbitrary points exist provided the Jacobian integral is smaller than the one given for some equilibrium point of the uncontrolled system. The structure of optimal controls is addressed in section 3, refining the results of [16]. In particular the role of peri- and apocenters with respect to global bounds on the number of switchings of the control is emphasized in connection with averaging of the system. The system has indeed two parameters: the bound of the control, ε, which can be taken very small with low-thrust applications in mind [31], thus leading to averaging, and the ratio of masses, μ, on which a continuation *à la Poincaré* may be performed to embed the two-body problem into a three-body one. This idea is the key to solve the problem as explained in section 4 where continuations both with respect to μ and ε are considered. The peculiarity of homotopy maps in optimal control is then illustrated in a simple framework in relation to second order optimality conditions.

2. Controllability. The drift has five equilibrium points, $L_1(\mu),\dots,L_5(\mu)$, that are known as *Lagrange points* [35] and whose position depends on μ. The points L_1, L_2, and L_3 are the *collinear* or *Euler points*, and the points L_4 and L_5 form two equilateral triangles with $-\mu$ and $1-\mu$. When $\mu = 0$, $L_1 = L_2 = 1$, $L_3 = -1$, and $L_4 = \exp(i\pi/3)$, $L_5 = -\exp(i\pi/3)$. (All belong to $\mathbf{S}^1$ which is a continuum of equilibrium points in this particular case.) The Jacobian J_μ is the only first integral of the nonintegrable uncontrolled system. Every level set $\{J_\mu = j\}$ projects onto $\Omega_\mu(q) + j = |\dot{q}|^2/2 \geq 0$ in the (q_1,q_2)-space, defining the Hill regions where the free motion has to take place (see Figure 1). Let $j_i(\mu) := J_\mu(L_i(\mu))$, $i = 1,\dots,5$ denote the Jacobian constants of these points. For $\mu \in (0,1)$, $j_2 < j_1 < j_3 < j_4 = j_5$. (These values all degenerate to $-3/2$ when μ goes to zero.) The open subset $\{x \in X_\mu \mid J_\mu(x) < j_1(\mu)\}$ has two connected components, and we denote by X_μ^1 the component containing $L_2(\mu)$ (see Figure 2). The result below essentially asserts that controllability for the restricted three-body problem holds provided the Jacobian is less than the Jacobian at L_1, emphasizing the role of Lagrange points. (Regarding the role of L_2, see the numerical results in section 4.)

THEOREM 1. *For any $\mu \in (0,1)$, for any positive ε, the circular restricted three-body problem is controllable on X_μ^1.*

We postpone the proof to the end of the section and first recall some basic facts needed to assert controllability.

Consider a smooth[1] control-affine system on a manifold X,

$$\dot{x}(t) = F_0(x(t)) + \sum_{i=1}^{m} u_i F_i(x(t)), \quad u(t) \in U \subset \mathbf{R}^m,$$

such that U is a neighborhood of the origin. The attainable set (by piecewise constant controls) from $x_0 \in X$ depends only on the drift, F_0, and on the distribution $\mathscr{D}$

[1] That is $\mathscr{C}^\infty$-smooth.

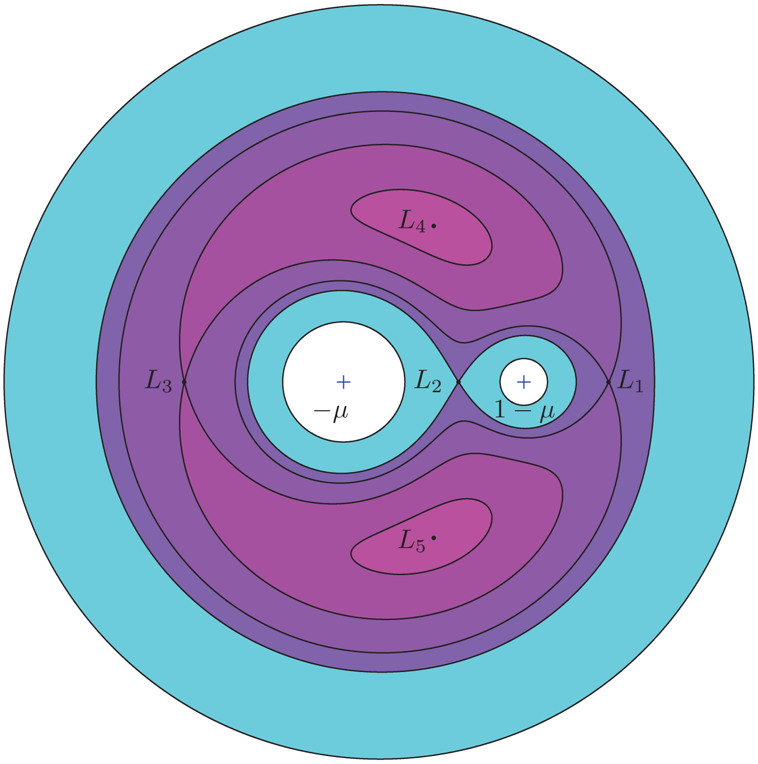

FIG. 1. *Lagrange points and associated Hill regions, $\mu \in (0,1)$. The forbidden regions of motion (complementary to the Hill regions) monotonically decrease as the Jacobian constant tends to $J_\mu(L_4) = J_\mu(L_5)$ (then disappear past this value) and opens up past $J_\mu(L_1)$.*

spanned by the vector fields $F_1, \dots, F_m$. It is the set of points obtained by compositions of flows,

$$e^{t_p G_p} \circ \cdots \circ e^{t_1 G_1}(x_0), \quad G_i \in F_0 + \mathscr{D}, \quad t_i \geq 0,$$

with t_i small enough for the composition to be defined. Now, if $\mathscr{F}$ is an arbitrary subsheaf of $\mathscr{C}^\infty$ vector fields on X, assuming for simplicity all $F \in \mathscr{F}$ complete, define the subgroup $\mathscr{G}$ of the diffeomorphisms of X generated by the one parameter subgroups $\exp tF$, $t \in \mathbf{R}$, $F \in \mathscr{F}$. According to the orbit theorem [1, 33], the orbit of $\mathscr{G}$ through x_0 is an immersed submanifold of X whose tangent space is

$$T_x\mathscr{G}(x_0) = \mathrm{Span}_x\{(\mathrm{Ad}\,\varphi)F,\ \varphi \in \mathscr{G},\ F \in \mathscr{F}\}.$$

In coordinates,

$$(\mathrm{Ad}\,\varphi)F_{|x} = [\varphi'(x)]^{-1}F(\varphi(x)), \quad x \in X.$$

Restricting to the $\mathscr{C}^\omega$-category, the adjoint action and the exponential commute in the sense that for arbitrary vector fields F, G,

$$(\mathrm{Ad}\,e^{tF})G_{|x} = (e^{t\,\mathrm{ad}F})G_{|x} = \sum_{n\geq 0} \frac{t^n}{n!}(\mathrm{ad}^n F)G_{|x}, \quad x \in X,$$

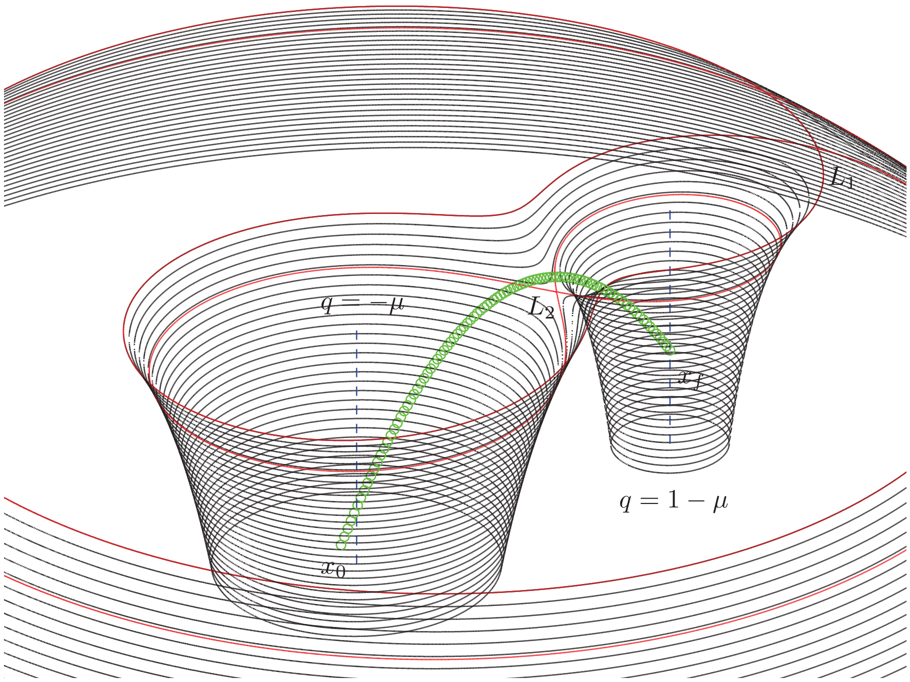

FIG. 2. *Projection of the open submanifold X^1_μ in the (q_1, q_2, J_μ)-space. The boundary of the volume is an apparent contour generated by the projection. It is the* zero velocity *set. Above each interior point there is an $\mathbf{S}^1$-fiber corresponding to the argument of $\dot{q}$. For $\mu \in (0,1)$, $j_2 < j_1$ and X^1_μ is connex. It is necessary that J_μ becomes greater than j_2 to make the transfer from x_0 to x_f. This strategy is observed on time minimum trajectories which pass in the neighborhood of the L_2 point.*

where $(\operatorname{ad} F)G$ is the Lie bracket, $[F,G] = F\cdot G - G\cdot F$ and recursively for ad^n. In this case, the orbit theorem then simply reads

$$T_x\mathscr{G}(x_0) = \operatorname{Lie}_x\mathscr{F}.$$

If the vector fields are not complete, $\mathscr{G}$ is just a pseudogroup [24], but the conclusion of the orbit theorem—which is local—is preserved. Coming back to control-affine systems, assuming that

$$\text{(1)} \qquad \operatorname{Lie}_x\{F_0, F_1, \ldots, F_m\} = T_xX, \quad x \in X,$$

and that the drift F_0 is recurrent[2] it can be proved that compositions with the flow of $-F_0$ can be added when computing the attainable set (see [22]). So the attainable set is equal to the orbit of the pseudogroup associated with $\{F_0, F_1, \ldots, F_m\}$. Because of the orbit theorem and of (1), this orbit has to be the whole manifold (supposed to be connected). In the control-affine three-body case, the rank condition holds, as is clear from the following.

LEMMA 1. *A second order controlled system on $\mathbf{R}^m$,*

$$\ddot{q}(t) + g(q(t), \dot{q}(t)) = u(t),$$

[2]Given a vector field F, a point $x \in X$ is recurrent or *positively Poisson stable* for F if for any neighborhood V of x, for any positive T, there is $t \geq T$ such that $\exp tF(x)$ is defined and belongs to V. The vector field itself is said to be recurrent when it has a dense subset of recurrent points.

is a control-affine system on $\mathbf{R}^{2m}$ *with an involutive distribution* $\mathscr{D}$ *and a drift* F_0 *such that* $\{F_1, \ldots, F_m, [F_0, F_1], \ldots, [F_0, F_m]\}$ *has maximum rank.*

Proof. As a first order system,

$$F_0(q, \dot{q}) = \dot{q}_1 \frac{\partial}{\partial q_1} + \cdots + \dot{q}_m \frac{\partial}{\partial q_m} - g_1(q, \dot{q}) \frac{\partial}{\partial \dot{q}_1} - \cdots - g_m(q, \dot{q}) \frac{\partial}{\partial \dot{q}_m}$$

and $F_i = \partial/\partial \dot{q}_i$ (so $\mathscr{D}$ is clearly involutive). Then

$$[F_0, F_i] = -\partial/\partial q_i \mod \mathscr{D}$$

and the rank is maximum. ∎

Applying the lemma to the planar three-body problem ($m = 2$), one obtains controllability in the following way.

Proof of Theorem 1. Let x_0, x_f in X_μ^1, and let j be strictly smaller than the Jacobian constants of both endpoints. Set $\widetilde{X}_\mu^1 := X_\mu^1 \cap \{J_\mu > j\}$. Outside a subset of zero measure associated with initial conditions generating collisions ($q = -\mu$ or $1 - \mu$), the drift is a complete Hamiltonian vector field whose exponential is defined for all times and is a volume preserving bijection in (q, p) coordinates. By definition, $\widetilde{X}_\mu^1$ which is a union of level sets of the Hamiltonian J_μ is invariant with respect to the exponential. For $x = (q, p) \in \widetilde{X}_\mu^1$,

$$j + \Omega_\mu(q) < \frac{1}{2}|p - iq|^2 < j_1(\mu) + \Omega_\mu(q).$$

Then, for a fixed q the volume of the q-section of $\widetilde{X}_\mu^1$ is bounded by $2\pi(j_1(\mu) - j)$ as is clear integrating with respect to $\mathrm{d}p_1 \wedge \mathrm{d}p_2 = \rho\, \mathrm{d}\rho \wedge \mathrm{d}\alpha$ (set $p - iq =: \rho \exp(i\alpha)$). Since the projection on the (q_1, q_2)-space of $\widetilde{X}_\mu^1$ is also bounded, the $\mathrm{d}q \wedge \mathrm{d}p$-measure of $\widetilde{X}_\mu^1$ is finite (Fubini). We conclude as in [32] that almost every point of $\widetilde{X}_\mu^1$ is recurrent by Poincaré's theorem. Controllability on $\widetilde{X}_\mu^1$ follows and implies the existence of a trajectory joining x_0 to x_f, which in turn implies controllability on X_μ^1. ∎

Remark 1. In the two-body case, $\mu = 0$, controllability still holds on X_0^1, the bounded component of $\{J_0 < j_1(0) = -3/2\}$. Each section of X_0^1 by a level set $\{J_0 = j\}$ is a pointed disk containing bounded, hence periodic, trajectories of the uncontrolled system. The energy is negative but, as $J_0 = E - C$, X_0^1 contains direct ($C > 0$), retrograde ($C < 0$), and collision orbits ($C = 0$). In contrast, the controllability result in [16] was obtained on the elliptic domain, $X_0 \cap \{E < 0,\ C > 0\}$, using the periodicity of the drift and excluding collisions. (A sign then had to be imposed on the momentum so that the manifold would be connex.)

Remark 2. In order to obtain an existence result for minimum time, one has to prove that minimizing trajectories remain into a fixed compact (which depends on the prescribed boundary conditions). Then, from any minimizing sequence one can extract a converging subsequence whose limit is an admissible trajectory by virtue of the convexity of the velocity field $\{F_0(x) + \varepsilon u_1 F_1(x) + u_2 F_2(x),\ |u| \leq 1\}$ for all $x \in X$ (Filippov theorem). One of the difficulties due to collisions is so to give bounds on the distance to the singularities, $-\mu$ and $1 - \mu$.

3. Singularities of the extremal flow. Let $u : [0, t_f] \to \mathbf{R}^2$ be a measurable time-minimizing control of the control-affine system

$$\dot{x}(t) = F_0(x(t)) + u_1 F_1(x(t)) + u_2 F_2(x(t)), \quad u_1^2(t) + u_2^2(t) \leq 1, \tag{2}$$

defined on a manifold X of dimension four. Let x denote the associated Lipschitz trajectory. Pontryagin's maximum principle [30] asserts that x is the projection of a Lipschitzian function, $z = (x, p) : [0, t_f] \to T^*X \backslash 0$, valued in the cotangent bundle minus the null section. The triple (x, u, p) is the reference *extremal*. In coordinates, there is a nonpositive scalar p^0 such that, a.e. on $[0, t_f]$,

$$\dot{x}(t) = \frac{\partial H}{\partial p}(x(t), u(t), p(t)), \quad \dot{p}(t) = -\frac{\partial H}{\partial x}(x(t), u(t), p(t))$$

with Hamiltonian $H(x, u, p) := p^0 + H_0(x, p) + u_1 H_1(x, p) + u_2 H_2(x, p)$ and Hamiltonian lifts $H_i(x, p) := \langle p, F_i(x) \rangle$, $i = 0, 1, 2$. Moreover, the maximization condition holds a.e.,

$$H(x(t), u(t), p(t)) = \max_{|v| \le 1} H(x(t), v, p(t)).$$

As a result, H is a.e. equal to a constant (zero here because the final time is free) along (x, u, p), and

$$u(t) = \frac{\psi(t)}{|\psi(t)|}$$

whenever the *switching function* $\psi(t) := (H_1, H_2)(x(t), p(t))$ does not vanish. The *switching surface* is

$$\Sigma := \{(x, p) \in T^*X \mid H_1(x, p) = H_2(x, p) = 0\},$$

and the crux for regularity is to study contacts (*switching points*) with Σ since outside the surface extremals are smooth. Extremals along which ψ does not vanish are *bang* extremals (denoted γ_b), while those on which ψ is identically zero are *singular* ones (denoted γ_s). We use the notation $F_{ij} := [F_i, F_j]$ (resp., $H_{ij} := \{H_i, H_j\}$) for Lie (resp., Poisson[3]) brackets of vector fields (resp., Hamiltonian lifts of these). The following analysis refines the one in [8, 16] using the tools of [10, 23].

We assume the following.

(i) $D(x) := \det(F_1(x), F_2(x), F_{01}(x), F_{02}(x)) \neq 0$, $x \in X$.

This assumption, which in particular implies that the span of F_1 and F_2 is of constant rank two, so that Σ is an embedded codimension two submanifold of the cotangent bundle, is geometric in the following sense. Let $\mathscr{D}$ be a rank 2 distribution over the four-dimensional manifold X (subbundle of TX with fibers of constant dimension two). Equipped with a Riemannian tensor g, $(\mathscr{D}, g)$ defines a sub-Riemannian structure [3]. Given a vector field F_0 over X, consider the problem of finding the minimum time Lipschitz trajectories subject to

$$\dot{x}(t) = F_0(x(t)) + v, \quad v \in \mathscr{D}, \quad |v|_g := \sqrt{g_{x(t)}(v)} \le 1$$

that connect prescribed points of X. Given any local frame $\{F_1, F_2\}$ of $\mathscr{D}$ orthonormal with respect to g, this problem is reformulated as (2). Obviously, the previous assumption only depends on the distribution and on the drift (see also Lemma 3).

[3]The Poisson bracket of two smooth functions f, g on T^*X is $\{f, g\} := \sum_i^n \partial_{x_i} g \, \partial_{p_i} f - \partial_{x_i} f \, \partial_{p_i} g$, $n = \dim X$. In particular, the Poisson bracket of lifts of vector fields F_i, F_j is the lift of their Lie bracket, $\{H_i, H_j\} = H_{ij}$.

LEMMA 2. *If a singular extremal passes through $z_0 \in \Sigma$, then $H_{12}(z_0) \neq 0$.*

Proof. The switching function is Lipschitz and almost everywhere,

$$\dot\psi_1(t) = H_{01}(z(t)) - u_2(t) H_{12}(z(t)), \tag{3}$$

$$\dot\psi_2(t) = H_{02}(z(t)) + u_1(t) H_{12}(z(t)). \tag{4}$$

Assume by contradiction $H_{12}(z_0) = 0$. Along a singular extremal, ψ vanishes identically and so does $\dot\psi$. Then H_1, H_2, H_{01}, and H_{02} vanish at $z_0 = (x_0, p_0)$, which implies $p_0 = 0$ because of assumption (i). This is impossible along a minimum time extremal. ∎

In the neighborhood of $z_0 \in \Sigma$ such that $H_{12}(z_0) \neq 0$, the following dynamical feedback is well defined:

$$u_s(z) := \frac{1}{H_{12}(z)}(-H_{02}, H_{01})(z). \tag{5}$$

(Compare [17], where, on the contrary, singular extremals are studied in the involutive case.) Plugging this control into H sets up a new Hamiltonian,

$$H_s(z) := H(z, u_s(z)) = p^0 + H_0(z) + u_{s,1}(z) H_1(z) + u_{s,2}(z) H_2(z).$$

PROPOSITION 1. *Let $z_0 \in \Sigma$, $H_{12}(z_0) \neq 0$; there is exactly one singular extremal passing through z_0, and it is defined by the flow of H_s.*

Proof. First we show that Σ is invariant with respect to the flow of H_s. Let $z_0 \in \Sigma$, $H_{12}(z_0) \neq 0$, and let z_s be the associated integral curve of H_s through it. Let $\varphi := (H_1, H_2) \circ z_s$; then φ is smooth and

$$\begin{aligned}\dot\varphi_1(t) &= \{H_s, H_1\}(z_s(t)) \\ &= \underbrace{H_{01} - u_{s,2} H_{12} \mid_{z_s(t)}}_{0} + \{u_{s,1}, H_1\} H_1 + \{u_{s,2}, H_1\} H_2 \mid_{z_s(t)},\end{aligned}$$

and similarly for $\dot\varphi_2$, so $\dot\varphi(t) = A(t)\,\varphi(t)$ with

$$A(t) := \begin{bmatrix} \{u_{s,1}, H_1\} & \{u_{s,2}, H_1\} \\ \{u_{s,1}, H_2\} & \{u_{s,1}, H_2\} \end{bmatrix} (z_s(t)).$$

Since $\varphi(0) = (H_1, H_2)(z_0) = (0, 0)$, φ is identically zero and z_s remains on Σ. Now,

$$H_s'(z) = \frac{\partial H}{\partial z}(z, u_s(z)) + \frac{\partial H}{\partial u}(z, u_s(z)) u_s'(z), \quad \frac{\partial H}{\partial u}(z, u) = (H_1, H_2)(z),$$

so $\overrightarrow{H}_s(z_s(t)) = \overrightarrow{H}(z_s(t), u_s(z_s(t)))$ as $\partial H/\partial u$ vanishes along z_s, and $(z_s, u_s \circ z_s)$ is extremal. ∎

Consider the stratification of $\Sigma = \Sigma_- \cup \Sigma_0 \cup \Sigma_+$, where

$$\begin{aligned}\Sigma_- &:= \{z \in \Sigma \mid H_{12}^2(z) < H_{01}^2(z) + H_{02}^2(z)\}, \\ \Sigma_0 &:= \{z \in \Sigma \mid H_{12}^2(z) = H_{01}^2(z) + H_{02}^2(z)\}, \\ \Sigma_+ &:= \{z \in \Sigma \mid H_{12}^2(z) > H_{01}^2(z) + H_{02}^2(z)\}.\end{aligned}$$

We use a nilpotentatization to study the behavior of bang extremals in the neighborhood of points in $\Sigma_- \cup \Sigma_+$.

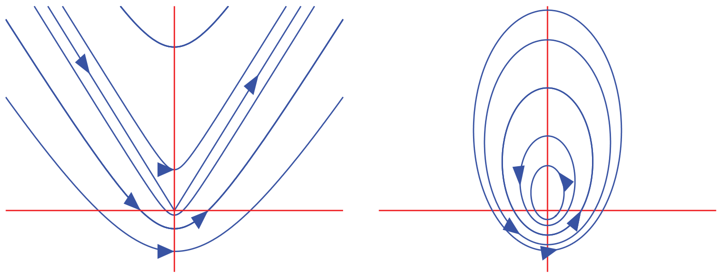

FIG. 3. *Phase portraits of the switching function under assumption* (i). *For $z_0 \in \Sigma_-$, the half-line $\theta = \pi - \theta_b(z_0)$ (resp., $\theta = \theta_b(z_0)$) goes to the origin (resp., departs from the origin).*

In the nilpotent approximation around a point $z_0 = (x_0, p_0) \in \Sigma\backslash 0$, Poisson brackets of length greater or equal to three vanish; since the time derivatives of the length two brackets only involve such brackets, H_{01}, H_{02}, and H_{12} are constant in this approximation. Under assumption (i), $\{F_1, F_2, F_{01}, F_{02}\}$ form a frame so $(H_{01}, H_{02})(z_0) \neq (0,0)$ since p_0 would otherwise be zero. Set

$$H_{01}(z_0) =: a_1, \quad H_{02}(z_0) =: a_2, \quad H_{12}(z_0) =: b$$

with $(a_1, a_2) \neq (0,0)$. Making a polar blowing up $(\psi_1, \psi_2) = (\rho\cos\theta, \rho\sin\theta)$, the differential equation for the switching function in the nilpotent approximation in the neighborhood of z_0 reads

$$\dot\rho(t) = a_1\cos\theta(t) + a_2\sin\theta(t), \quad \rho(t)\dot\theta(t) = b - a_1\sin\theta(t) + a_2\cos\theta(t),$$

which up to some rotation and rescaling can be normalized to

$$\dot\rho(t) = \cos\theta(t), \quad \rho(t)\dot\theta(t) = c - \sin\theta(t)$$

with $c := b/\sqrt{a_1^2 + a_2^2}$. This system is integrated according to

$$\rho(\theta) = \rho(\theta(0))\left|\frac{c - \sin\theta(0)}{c - \sin\theta}\right|,$$

whence the phase portraits in Figure 3 for $|c| < 1$ ($z_0 \in \Sigma_-$) and $|c| > 1$ ($z_0 \in \Sigma_+$). When $|c| < 1$, the origin is reached along $\theta = \pi - \theta_b(z_0)$ ($\dot\rho < 0$) and departs from it along $\theta = \theta_b(z_0)$ ($\dot\rho > 0$), where

$$\theta_b(z_0) := \arcsin\frac{H_{12}}{\sqrt{H_{01}^2 + H_{02}^2}}(z_0).$$

To summarize, the following is noted.

THEOREM 2. *Let $z_0 \in \Sigma_-$; every extremal is locally of the form $\gamma_b\gamma_s\gamma_b$ (γ_s empty if $H_{12}(z_0) = 0$); every admissible extremal is locally the concatenation of at most two bang arcs. Let $z_0 \in \Sigma_+$; every extremal is locally bang or singular, and every optimal extremal is locally bang. Optimal singular extremals are given by the flow of H_s and contained in Σ_0 (saturating).*

Proof. A singular extremal passing through $z_0 \in \Sigma_-$ cannot be admissible since, if $H_{12}(z_0) \neq 0$, the singular control is well defined but $|u_s(z_0)| > 1$. A singular extremal passing through $z_0 \in \Sigma_+$ is admissible since $|u_s(z_0)| < 1$ but cannot be optimal: As it is interior to the constraint, it must satisfy the Goh second order necessary condition [1],

$$\left\{\frac{\partial H}{\partial u_1}, \frac{\partial H}{\partial u_2}\right\}(z_0, u_s(z_0)) = H_{12}(z_0) = 0,$$

which is excluded. In the neighborhood of Σ_+ points, the connection between bang and singular extremals is not possible according to the phase portrait in the nilpotent approximation. ◻

An example of saturating singular control is provided by the following nilpotent system (compare [8]). Let

$$F_0(x) = (1+x_3)\frac{\partial}{\partial x_1} + x_4\frac{\partial}{\partial x_2}, \quad F_1(x) = x_4\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3}, \quad F_2(x) = \frac{\partial}{\partial x_4}\cdot$$

One checks that $H_{12} = H_{01} = -p_1$, $H_{02} = -p_2$, so Σ_0 is defined by $x_4 p_1 + p_3 = 0$, $p_4 = 0$, and $p_2 = 0$. The singular control is $u_s(z) = (-p_2/p_1, 1)$ and

$$H_s(x,p) = (1+x_3)p_1 - \frac{p_2 p_3}{p_1} + p_4.$$

Through $z_0 \in \Sigma_0$ such that $x_{40} = 0$ passes the singular extremal

$$x_1(t) = (1+x_{30})t + x_{10}, \quad x_2(t) = \frac{t^2}{2} + x_{20}, \quad x_3(t) = x_{30}, \quad x_4(t) = t,$$
$$p_1(t) = p_{10} \neq 0, \quad p_2(t) = 0, \quad p_3(t) = -p_{10}\, t, \quad p_4(t) = 0,$$

associated with $u_s = (0,1)$.

Let us now define

$$D_1(x) := \det(F_1(x), F_2(x), F_{12}(x), F_{02}(x)),$$
$$D_2(x) := \det(F_1(x), F_2(x), F_{01}(x), F_{12}(x))$$

to strengthen and replace assumption (i) by the following:

(i′) $D_1^2(x) + D_2^2(x) < D^2(x)$, $x \in X$.

This assumption only depends on the sub-Riemannian structure and the drift.

LEMMA 3. *Assumption* (i′) *is independent of a particular choice of orthonormal frame on* $(\mathscr{D}, g)$.

Proof. Let $\{F_1, F_2\}$ and $\{\hat{F}_1, \hat{F}_2\}$ be two orthonormal bases in the neighborhood of some point on x. There exists a smooth function θ such that in coordinates,

$$\hat{F}_1(x) = \cos\theta(x) F_1(x) + \sin\theta(x) F_2(x),$$
$$\hat{F}_2(x) = \varepsilon(-\sin\theta(x) F_1(x) + \cos\theta(x) F_2(x))$$

with $\varepsilon = \pm 1$. One can restrict to $\varepsilon = 1$ and use the facts

$$[F, \beta G] = [F, G] \mod \mathbf{R}G, \quad [\alpha F, \beta G] = [F, G] \mod \operatorname{Span}\{F, G\}$$

to check that $\hat{D}_1 = D_1$, $\hat{D}_2 = D_2$, and $\hat{D} = D$, where

$$\hat{D}_1(x) := \det(\hat{F}_1(x), \hat{F}_2(x), [\hat{F}_1, \hat{F}_2](x), [\hat{F}_0, \hat{F}_2](x)), \quad \text{etc.} \qquad ◻$$

Remark 3. When the distribution is involutive (which we shall assume later), $D_1 = D_2 = 0$ and with obvious notation (i′) simply asserts that $\mathscr{D} + [F_0, \mathscr{D}]$ has full rank—this is assumption (i). Assumption (i′) can so be interpretated as "bounding the nonholonomy" of $\mathscr{D}$ with respect to F_0.

COROLLARY 1. *Every extremal is locally of the form $\gamma_b\gamma_s\gamma_b$ (γ_s possibly empty); for such a sequence, the total angle switch of the control between the entry point into Σ, z_0, and the exit point z_0' is $\theta_b(z_0) + \theta_b(z_0') + \pi$. Admissible extremals are the concatenation of finitely many bang arcs.*

Proof. Computing

$$\begin{aligned}&\det(F_1(x), F_2(x), F_{01}(x) - u_2F_{12}(x), F_{02}(x) + u_1F_{12}(x))\\&\quad = D(x) - u_2D_1(x) + u_1D_2(x), \quad x \in X, \quad u \in \mathbf{R}^2,\end{aligned}$$

for $|u| \leq 1$, $\{F_1, F_2, F_{01} - u_2F_{12}, F_{02} + u_1F_{12}\}$ form a frame by virtue of (i′). Let $z_0 = (x_0, p_0)$ belong to $\Sigma\backslash 0$. For an arbitrary $u \in \mathbf{R}^2$, $|u| \leq 1$,

$$\begin{bmatrix} \langle p_0, F_{01}(x_0)\rangle \\ \langle p_0, F_{02}(x_0)\rangle \end{bmatrix} + \langle p_0, F_{12}(x_0)\rangle \begin{bmatrix} -u_2 \\ u_1 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

since p_0 would otherwise be zero. As a result

$$H_{12}^2(z_0) < H_{01}^2(z_0) + H_{02}^2(z_0),$$

so $\Sigma\backslash 0 = \Sigma_-$ and the local structure follows from the previous theorem. If the singular arc is not empty, by virtue of (5) the angle switch at the bang-singular junction is $(\varepsilon\pi/2 + \theta_0) - (\pi - \theta_b(z_0) + \theta_0)$ with $\varepsilon := \operatorname{sign} H_{12}(z_0)$, θ_0 being the argument of $(H_{01}, H_{02})(z_0) \neq (0, 0)$. Similarly, the angle switch at the singular-bang junction is $(\theta_b(z_0') + \theta_0') - (\varepsilon'\pi/2 + \theta_0')$. As H_{12} does not vanish along the singular arc, $\varepsilon = \varepsilon'$, whence the result. If the singular arc is empty (notably when $H_{12}(z_0) = 0$), the angle switch is $(\theta_b(z_0) + \theta_0) - (\pi - \theta_b(z_0) + \theta_0)$ and the formula still holds with $z_0 = z_0'$. ∎

Assuming moreover that

(ii) $\mathscr{D}$ is involutive,

we get the following.

COROLLARY 2 (see [16]). *The switching function is continuously differentiable, and every extremal is locally bang-bang with switchings of angle π ("π-singularities").*

Proof. For $z_0 \in \Sigma\backslash 0$, one has $H_{12}(z_0) = 0$ because of (ii), so no singular arc passes through a switching point; $\theta_b(z_0) = \pi$ and the angle switch is π. As the bracket H_{12} vanishes at switching points, $\dot\psi$ is continuous as observed from (3)–(4). ∎

In order to give a global bound on the number of switchings, we finally add the following assumption:

(iii) $F_0 \notin \operatorname{Span}\{F_1, F_2, F_{01}\}$.

Then we define

$$\Sigma_1 := \Sigma \cap \{(x, p) \in T^*X \mid F_0(x) \in \operatorname{Span}_x\{F_1, F_2, F_{02}\}\}.$$

Remark 4. Properties (i′) + (ii) are equivalent to (i) + (ii). In contrast to (i), (i′) and (ii), property (iii) (and Σ_1) does depend on the particular choice of orthonormal basis of $\mathscr{D}$.

THEOREM 3 (see [16]). *In the normal case, there cannot be consecutive switchings in Σ_1. In particular, if $\Sigma = \Sigma_1$, any normal optimal control has at most one switching which is a π-singularity.*

Proof. According to (i), there are continous scalar functions λ_1, λ_2 on X such that

$$F_0 = \lambda_1 F_{01} + \lambda_2 F_{02} \mod \mathscr{D}.$$

At a switching point $z_0 = (x(t_0), p(t_0)) \in \Sigma_1 \backslash 0$,

$$H(x(t_0), u(t_0), p(t_0)) = -p^0 = \lambda_2(x(t_0))H_{02}(z_0) = \lambda_2(x(t_0))\dot{\psi}_2(t_0).$$

In the normal case, $p^0 < 0$ so

$$\lambda_2(x(t_1))\dot{\psi}_2(t_1)\lambda_2(x(t_2))\dot{\psi}_2(t_2) = (p^0)^2 > 0$$

if we assume that $t_1 < t_2$ are consecutive such switching times. Because of (iii), λ_2 never vanishes; by continuity its sign is constant so $\dot{\psi}_2(t_1)\dot{\psi}_2(t_2) > 0$, and the contradiction follows. ▯

Thanks to Lemma 1, these results apply to the problem under consideration: The circular restricted three-body has bang-bang time-minimizing controls with finitely many π-singularities. The study of such singularities is important since in practice the rotation speed of the thrust is limited. A remarkably simple geometric interpretation of Σ_1 is obtained when restricting to a two-body system, $\mu = 0$. This case is not only important in itself as it also corresponds to the initial and final phases of a typical three-body low-thrust transfer. Such a trajectory resembles a heteroclinic trajectory connecting periodic orbits around each one of the primaries (see section 4). We use for the analysis the geometric coordinates and the radial-orthoradial frame introduced in section 1. We assume the eccentricity positive, $0 < e < 1$, to obviate the singularity of these coordinates at circular orbits. With $x = (n, e, \theta, l) \in \mathbf{R}_+^* \times (0,1) \times \mathbf{S}^1 \times \mathbf{R}$ (and $\tau = l - \theta$), $p = (p_n, p_e, p_\theta, p_l) \in (\mathbf{R}^4)^*$, we set

$$\alpha := -\frac{3n}{1-e^2}\, p_n, \quad \beta := p_e, \quad \gamma := \frac{p_\theta}{e}, \quad c := \cos\tau, \quad s := \sin\tau,$$

so Σ is defined by the following algebraic system:

$$\begin{aligned} \alpha e s + \beta s - \gamma c &= 0, \\ \alpha(1+ec) + \beta\left(c + \frac{e+c}{1+ec}\right) + \gamma\left(s + \frac{s}{1+ec}\right) &= 0, \\ c^2 + s^2 &= 1. \end{aligned}$$

LEMMA 4. *The switching surface Σ for $\mu = 0$ is stratified as follows:*

(a) *If $\gamma = 0$, either $(\alpha, \beta) = (0,0)$ or $s = 0$ and α, β belong to the union of the two distinct lines*

$$(1+e)\alpha + 2\beta = 0, \quad (1-e)\alpha - 2\beta = 0.$$

(b) *If $\gamma \neq 0$, s is not zero and α, β are uniquely determined.*

Proof. The algebraic system in α, β

$$\begin{aligned} \alpha e s + \beta s &= \gamma c, \\ \alpha(1+ec) + \beta\left(c + \frac{e+c}{1+ec}\right) &= -\gamma\left(s + \frac{s}{1+ec}\right) \end{aligned}$$

has determinant

$$\begin{vmatrix} es & s \\ 1+ec & c+\dfrac{e+c}{1+ec} \end{vmatrix} = -\frac{s(1-e^2)}{1+ec}\,,$$

whence the result. ∎

PROPOSITION 2. *In the two-body case $\mu = 0$, the subset Σ_1 is the stratum $\{s = 0\}$ of Σ. Accordingly,*

$$\Sigma \cap \{p_\theta = 0\} = \Sigma_1 \cup \{(p_n, p_e, p_\theta) = (0, 0, 0)\}.$$

Proof. Using the fact that $F_0 \in \mathrm{Span}\{\partial/\partial l\}$ and that for smooth functions f, g

$$[fF_0, g\widetilde{F}_2] = fg[F_0, \widetilde{F}_2] \mod \mathrm{Span}\{F_0, \widetilde{F}_2\},$$

the condition $\det(F_0, \widetilde{F}_1, \widetilde{F}_2, [F_0, \widetilde{F}_2]) = 0$ is equivalent to

$$\begin{vmatrix} 0 & es & 1+ec & -es \\ 0 & s & c+\dfrac{e+c}{1+ec} & -s-\dfrac{s(1-e^2)}{(1+ec)^2} \\ 0 & -c & s+\dfrac{s}{1+ec} & c+\dfrac{e+c}{(1+ec)^2} \\ 1 & 0 & 0 & 0 \end{vmatrix} = 0,$$

that is, to

$$s = 0 \quad \text{or} \quad e(1+ec) = 0,$$

so the conclusion follows. ∎

Geometrically, $s = \sin\tau = 0$ (that is, $l - \theta = 0 \mod \pi$) corresponds to peri- and apocenters. The importance of the stratum $\Sigma \cap \{p_\theta = 0\}$ comes from the analysis of the system when the control magnitude ε goes to zero (low-thrust case). When the control is small, not only the time but also the angular length (longitude) needed to connect two Keplerian orbits in the two-body model becomes large. It thus makes sense to use averaging to analyze the behavior of extremal trajectories. In the case of the minimization of the L^2-norm of the control (*energy* minimization), this approach goes back to [19]. (See also [13] for a recent treatment of this question.) One first uses the fact that in the planar model the drift only acts on the longitude,

$$\dot{l}(t) = \omega(x(t)), \quad \omega(x) := \frac{nW^2}{(1-e^2)^{3/2}}\,, \quad W = 1 + e\cos\tau,$$

to set l as the new time. In this new parameterization, minimum time extremals are integral curves of the maximized Hamiltonian

$$\hat{H}(l, \hat{x}, \hat{p}) := \frac{p^0}{\omega(l, \hat{x})} + \varepsilon\sqrt{\hat{H}_1^2(l, \hat{x}, \hat{p}) + \hat{H}_2^2(l, \hat{x}, \hat{p})}$$

with $\hat{x} = (n, e, \theta)$, $\hat{p} = (p_n, p_e, p_\theta)$, and

$$\begin{aligned} &\hat{H}_1^2(l, \hat{x}, \hat{p}) + \hat{H}_2^2(l, \hat{x}, \hat{p}) \\ &= \frac{(1-e^2)^4}{n^{8/3}W^4}\left\{\frac{9n^2}{(1-e^2)^2}(1 + 2e\cos\tau + e^2)p_n^2\right. \\ &\qquad - \frac{12n}{1-e^2}(e+\cos\tau)p_np_e \\ &\qquad \left.+ \left[1 + \frac{2(e+\cos\tau)}{W}\cos\tau + \frac{(e+\cos\tau)^2}{W^2}\right]p_e^2\right\} + \cdots, \end{aligned}$$

where the dots indicate terms in p_θ^2, $p_n p_\theta$, and $p_e p_\theta$. The averaged Hamiltonian is

$$\overline{\hat{H}}(\hat{x},\hat{p}) := \frac{1}{2\pi}\int_0^{2\pi} \hat{H}(l,\hat{x},\hat{p})\,\mathrm{d}l. \tag{6}$$

PROPOSITION 3. *The argument of pericenter θ is a cyclic variable of the averaged system. On the stratum $\{p_\theta = 0\}$, the integral* (6) *is hyperelliptic of genus* 2.

Proof. As is clear from the whole dynamics in geometric coordinates (see section 1), only the difference $\tau = l - \theta$ appears in $\hat{H}$. Averaging with respect to l—or equivalently to τ—kills terms in θ and p_θ becomes a linear first integral. Then on $\{p_\theta = 0\}$

$$\sqrt{\hat{H}_1^2 + \hat{H}_2^2} = \frac{1}{W^3}\sqrt{R(\cos\tau)},$$

where R is a degree 3 polynomial with coefficients depending nonlinearly on $\hat{x}$ and quadratically on $\hat{p}$. Setting for instance $u = \cos\tau$ leads to

$$\int \frac{\mathrm{d}u}{1-u^2}\sqrt{(1-u^2)R(u)}$$

which is an integral parameterized by a genus 2 hyperelliptic curve [21]. ∎

For circular targets (which are of great practical importance in two or three-body control), the transversality condition is $p_\theta = 0$. The previous analysis then suggests that for small control magnitudes, p_θ should also remain small so that the switching structure will be close to the one on $\Sigma \cap \{p_\theta = 0\}$: Assuming there are no trivial switchings $(p_n, p_e, p_\theta) = 0$, one would get a global bound on the number of π-singularities (at most one). Nevertheless, one should notice that if p_θ is small but not zero, the switchings do not belong to Σ_1 according to Lemma 4 and Proposition 2. Moreover, when studying the convergence of the system toward the averaged one, one should take into accout the lack of regularity due to the radicand vanishing. For instance,

$$\int_0^{\pi/2} \sqrt{\sin^2 l + z^2}\,\mathrm{d}l = |z|E(-iz^{-1}),$$

where E is the complete elliptic integral of second kind. Such a function has a $z^2 \log|z|$ singularity at the origin and is not $\mathscr{C}^2$ (logarithmic branch on the second derivative). See [7] for results on averaging of such *fast oscillating systems*.

4. Homotopy. Consider a control problem with smooth data on an n-manifold X,

$$\dot{x}(t) = f(x(t), u(t)), \quad u(t) \in U,$$

with cost

$$\int_0^{t_f} f^0(x(t), u(t))\mathrm{d}t \to \min$$

and prescribed boundary conditions

$$x(0) = x_0, \quad x(t_f) = x_f.$$

For the sake of simplicity t_f is supposed to be fixed, but the analyis below can be made with appropriate changes for free final time as well. We also suppose that U is a manifold without boundary. In coordinates, this is assuming that u belongs to some open subset of $\mathbf{R}^m$, where m is the dimension of U. Regarding the results of the previous section, in the three-body case this amounts to assuming there are no π-singularities.

Let $\overline{u} : [0, t_f] \to U$ be a measurable and essentially bounded control, and let $\overline{x} : [0, t_f] \to X$ be the resulting trajectory. Pontryagin's maximum principle implies that there exists a nonpositive constant $\overline{p}^0$ and a Lipschitz covector function $\overline{p} : [0, t_f] \to (\mathbf{R}^n)^*$, not both zero, so that in coordinates on T^*X,

$$\dot{\overline{x}}(t) = \frac{\partial H}{\partial p}(\overline{x}(t), \overline{u}(t), \overline{p}(t)), \quad \dot{\overline{p}}(t) = -\frac{\partial H}{\partial x}(\overline{x}(t), \overline{u}(t), \overline{p}(t)),$$

and

$$H(\overline{x}(t), \overline{u}(t), \overline{p}(t)) = \max_{v \in U} H(\overline{x}(t), v, \overline{p}(t))$$

a.e. on $[0, t_f]$. Here,

$$H(x, u, p) := p^0 f^0(x, u) + \langle p, f(x, u) \rangle.$$

We assume that on a neighborhood of the extremal $(\overline{x}, \overline{p})$, the maximized Hamiltonian

$$(x, p) \mapsto \max_{v \in U} H(x, u, p)$$

is well defined and smooth; then $(\overline{x}, \overline{p})$ is an integral curve of the maximized function (see [1]), still denoted H (but now depending only on x and p). We finally make the Legendre regularity assumption that uniformly on $[0, t_f]$,

$$\nabla^2_{uu} H(\overline{x}(t), \overline{u}(t), \overline{p}(t)) \leq -\alpha I_m$$

for some $\alpha > 0$. As a consequence, there must exist in a neighborhood of the extremal a smooth implicit function $u(x, p)$ solving the first order necessary condition $\nabla_u H = 0$ such that $\overline{u}(t) = u(\overline{x}(t), \overline{p}(t))$ and $H(x, p) = H(x, u(x, p), p)$. Summarizing, $\overline{p}(0) \in (\mathbf{R}^n)^*$ is a zero of the *shooting function*[4]

$$p_0 \mapsto x(t_f, x_0, p_0) - x_f,$$

where the *exponential mapping*

$$\exp_{x_0} : (t, p_0) \mapsto x(t, x_0, p_0)$$

sends a given p_0 to the x-projection of the integral curve at t of the maximized Hamiltonian. Both functions are well defined and smooth on neighborhoods of $\overline{p}(0)$ and $(t_f, \overline{p}(0))$, respectively. A time t_c is *conjugate* to 0 along $(\overline{x}, \overline{p})$ whenever $\overline{p}(0)$ is a critical point of $p_0 \mapsto \exp(t_c, p_0)$. The critical value $x_c = \exp(t_c, \overline{p}(0))$ is the corresponding *conjugate point*. These notions are related to local necessary or sufficient second order conditions of optimality [1, 10]. Testing conjugacy is done in practice by a simple rank evaluation (see [9]; see also Figure 4).

The *endpoint mapping*

$$F_{t_f, x_0} : u \mapsto \hat{x}(t_f, x_0, u)$$

is well defined and smooth on a neighborhood in $\mathrm{L}^\infty([0, t_f], U)$ of (the class of) $\overline{u}$ and maps a control to the solution at t_f of the *augmented system* ($\hat{x} = (x^0, x)$)

$$\begin{aligned} \dot{x}^0(t) &= f^0(x(t), u(t)), \quad t \in [0, t_f] \quad \text{(a.e.)}, \\ \dot{x}(t) &= f(x(t), u(t)), \quad x^0(0) = 0, \quad x(0) = x_0. \end{aligned}$$

[4] A chart (O, φ) in the neighborhood of x_f has to be chosen, and the definition should read $p_0 \mapsto \varphi(x(t_f, x_0, p_0)) - \varphi(x_f)$. We may actually suppose that $\varphi(x_f) = 0$.

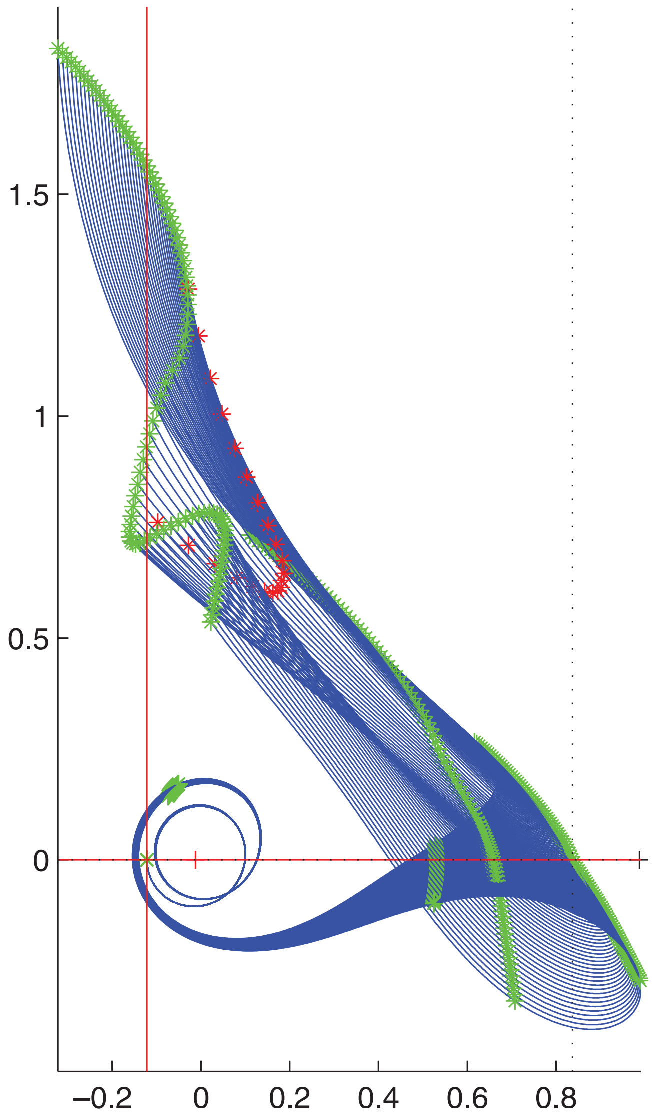

FIG. 4. *Conjugate point computation (rotating frame). Extremals (here projected on the* (q_1, q_2)*-space) from a circular orbit around the first primary toward the* L_2 *Lagrange point are extended beyond the target. Conjugate points, in red, appear after* t_f*, ensuring local optimality. Green dots indicate isocost (isotime) lines.*

The optimal control $\overline{u}$ must be a critical point[5] of the endpoint mapping: $\operatorname{Im} F'_{t_f,x_0}(\overline{u})$ has codimension in $\mathbf{R}^n$. If we assume that $\overline{u}$ is a corank one critical point and moreover that it is analytical (with analytical data for the problem as well), the absence of conjugate time in $(0, t_f)$ is necessary for L^∞-local optimality.[6] Conversely, replacing the corank one and analyticity conditions by the assumption that the extremal is

[5]This statement, weaker than the maximum principle, is obvious: Were the function a submersion at $\overline{u}$, it would be locally open and would send neighborhoods of $\overline{u}$ onto neighborhoods of the augmented state, $(\overline{x}^0(t_f), \overline{x}(t_f))$. This would contradict L^∞-local optimality.

[6]That is optimality on a neighborhood of $\overline{u}$ in $L^\infty([0, t_f], U)$.

normal ($p_0 < 0$), the absence of conjugate time on $(0, t_f]$ is sufficient for $\mathscr{C}^0$-local optimality.[7]

Having in mind these connections with second order conditions in optimal control (see also [11]) we consider a one-parameter smooth Hamiltonian,

$$H : \mathbf{R}^{2n} \times \mathbf{R} \ni (x, p, \lambda) \mapsto H(x, p, \lambda) \in \mathbf{R}.$$

Given a positive final time t_f and an initial condition x_0, we define the shooting-like[8] *homotopy function*[9]

$$h(p_0, \lambda) := x(t_f, x_0, p_0, \lambda)$$

that maps (p_0, λ) to the coordinate x of the solution at t_f of

$$\dot{x}(t) = \frac{\partial H}{\partial p}(x(t), p(t), \lambda), \quad \dot{p}(t) = -\frac{\partial H}{\partial x}(x(t), p(t), \lambda)$$

with initial conditions $x(0) = x_0$, $p(0) = p_0$. By restricting it if necessary, we may assume that its domain of definition, $\Omega \subset \mathbf{R}^{n+1}$, is open and made only of regular points of h so that

$$\operatorname{rank} h'(p_0, \lambda) = n, \quad (p_0, \lambda) \in \Omega.$$

As a consequence, the level set $\{h = 0\}$ is a one-dimensional submanifold of $\mathbf{R}^{n+1}$ called the *path of zeros*. Typically, one knows a zero of $h(., \lambda)$ for, say, $\lambda = 0$ and wants to follow this path to reach if possible a zero for a target value of the parameter, $\lambda = 1$. For any $c := (p_0, \lambda) \in \Omega$, $\dim \operatorname{Ker} h'(c) = 1$ so one can define the (tangent) vector $T(c)$ as being the unique—up to orientation—unit vector in the kernel. The orientation is chosen so that the nonvanishing determinant

$$\det \begin{bmatrix} h'(c) \\ {}^tT(c) \end{bmatrix}$$

has constant sign on each connected component of Ω. This provides a parameterization by arc length of the connected components of $\{h = 0\}$ which can be practically computed by integrating the following differential equation [2] (with $' = \mathrm{d}/\mathrm{d}s$):

$$c'(s) = T(c(s)), \quad c(0) = c_0 \in \{h = 0\}.$$

The aim is to classify each component up to diffeomorphisms, knowing that there are only two possibilities [27]: It is diffeomorphic either to $\mathbf{R}$ or to $\mathbf{S}^1$.

[7] This is the optimality of the trajectory among all admissible trajectories belonging to some neighborhood of $\overline{x}$.

[8] The target x_f is normalized to 0.

[9] For the use of homotopy in optimal control (in particular for motion planning), see also [18, 34], where the point of view is slightly different; the idea is to devise a *path lifting equation* to construct a path of zeros in the infinite dimensional set of controls. In this setting, the obstructions described at the end of the current section are translated as nondegeneracy and nonexplosion issues on the so-called Wazewski equation. Assuming more structure than we do on the dynamics (driftless affine control systems are considered), the authors are able to provide conditions in terms of the Lie algebra of controlled vector fields that overcome these difficulties. The emphasis is also on Galerkin procedures to solve the problem. To some extent, the situation is simpler in our case as considering the shooting function instead of the endpoint mapping restricts the problem to finite dimension.

In such a parameterization, a point $c(\overline{s}) = (p_0(\overline{s}), \lambda(\overline{s}))$ in $\{h = 0\}$ is a *turning point* [2] when $\lambda'(\overline{s}) = 0$. This is equivalent to saying that

$$\operatorname{rank} \frac{\partial x}{\partial p_0}(t_f, x_0, p_0(\overline{s}), \lambda(\overline{s})) = n - 1,$$

that is, to saying that t_f is a conjugate time for $\lambda = \lambda(\overline{s})$ (and that, respectively, $p_0(\overline{s})$ and $x(t_f, x_0, p_0(\overline{s}), \lambda(\overline{s}))$ are the corresponding critical and conjugate point). A turning point of order one, that is, such that $\lambda''(\overline{s}) \neq 0$, actually results in a change of variation on λ, hence the name. We define $\overline{c} = c(\overline{s}) \in \{h = 0\}$ to be a *first turning point* (along the path starting at $c(0)$) if for all $s \in [0, \overline{s})$, the curve $t \mapsto x(t, x_0, p_0(s), \lambda(s))$ has no conjugate time on $(0, t_f]$.

THEOREM 4. *Let $c(\overline{s}) \in \{h = 0\}$ be a first turning point of order one; then for $s > \overline{s}$, $|s - \overline{s}|$ small enough, there exist conjugate times in $(0, t_f)$.*

The next lemmas are necessary to prove this result. We first recall that at a corank one critical point $\overline{x}$ of a smooth function $g : \mathbf{R}^n \to \mathbf{R}^n$, one can define (up to a scalar) the *intrinsic second order derivative* [5] as

$$\overline{\mu} g''(\overline{x})_{|\operatorname{Ker} g'(\overline{x}) \times \operatorname{Ker} g'(\overline{x})} \in \mathscr{L}_2(\operatorname{Ker} g'(\overline{x}), \operatorname{Ker} g'(\overline{x}); \mathbf{R}) \simeq \mathbf{R},$$

where $\overline{\mu} \in (\mathbf{R}^n)^*$ is any nonzero covector with kernel $\operatorname{Im} g'(\overline{x})$. The critical point is said to be nondegenerate provided this quantity is not zero.

LEMMA 5. *The turning point $c(\overline{s})$ is of order one if and only if $p_0(\overline{s})$ is a nondegenerate corank one critical point of $p_0 \mapsto h(p_0, \lambda(\overline{s}))$.*

Proof. Differentiating twice $h(c(s)) = 0$, one gets

$$h''(c(\overline{s})) \cdot (c'(\overline{s}), c'(\overline{s})) + h'(c(\overline{s})) \cdot c''(\overline{s}) = 0.$$

As $c'(\overline{s}) = (p_0'(\overline{s}), 0)$, $p_0'(\overline{s})$ generates the kernel of $\partial h/\partial p_0(p_0(\overline{s}), \lambda(\overline{s}))$ and

$$\frac{\partial h}{\partial p_0}(c(\overline{s})) \cdot p_0''(\overline{s}) + \frac{\partial h}{\partial \lambda}(c(\overline{s})) \cdot \lambda''(\overline{s}) = -\frac{\partial^2 h}{\partial p_0^2}(c(\overline{s})) \cdot (p_0'(\overline{s}), p_0'(\overline{s})).$$

Multiplying both sides by any nonzero $\overline{\mu}$ whose kernel coincides with the image of $\partial h/\partial p_0(p_0(\overline{s}), \lambda(\overline{s}))$, one gets

$$\overline{\mu} \frac{\partial h}{\partial \lambda}(c(\overline{s})) \cdot \lambda''(\overline{s}) = -\overline{\mu} \frac{\partial^2 h}{\partial p_0^2}(c(\overline{s})) \cdot (p_0'(\overline{s}), p_0'(\overline{s})).$$

At the turning point $c(\overline{s}) \in \Omega$, $\partial h/\partial \lambda$ is transverse to the image of $\partial h/\partial p_0$ (regularity). So $\overline{\mu}\, \partial h/\partial \lambda(c(\overline{s})) \neq 0$, and $\lambda''(\overline{s}) = 0$ if and only if the intrinsic second derivative vanishes. ∎

Remark 5. The order one assumption thus puts some restriction on

$$\frac{\partial^2 x}{\partial p_0^2}(t_f, x_0, p_0(\overline{s}), \lambda(\overline{s})).$$

It was previously mentioned that the first order derivative with respect to p_0 of this function is connected with second order optimality conditions. Here we have a condition of order three.

LEMMA 6. *Let $\overline{x}$ be a corank one critical point of a smooth function $g : \mathbf{R}^n \to \mathbf{R}^n$. Then $\overline{x}$ is degenerate if and only if*

$$(\det g')'(\overline{x}) = 0 \quad \textit{on} \quad \operatorname{Ker} g'(\overline{x}).$$

Proof. Let $h \in \mathbf{R}^n$; one has

$$(\det g')'(\overline{x}) \cdot h = \operatorname{tr}(\widetilde{g'(\overline{x})} \cdot g''(\overline{x}) \cdot h).$$

Since $g'(\overline{x})$ is of rank $n-1$, one can find a nonzero vector $\overline{\xi} \in \operatorname{Ker} g'(\overline{x})$ (resp., covector $\overline{\mu}$ with kernel $\operatorname{Im} g'(\overline{x})$) such that for the adjugate matrix

$$\widetilde{g'(\overline{x})} = \overline{\xi}\overline{\mu}.$$

Thus,

$$(\det g')'(\overline{x}) \cdot h = \sum_{j=1}^{n} \operatorname{tr}\left(\overline{\xi}\overline{\mu}\frac{\partial g'}{\partial x_j}(\overline{x})\right) h_j = \sum_{j=1}^{n} \overline{\mu}\frac{\partial g'}{\partial x_j}(\overline{x})\overline{\xi} h_j = \overline{\mu} g''(\overline{x})(\overline{\xi}, h).$$

In particular, for $h = \overline{\xi} \in \operatorname{Ker} g'(\overline{x})$,

$$(\det g')'(\overline{x}) \cdot \overline{\xi} = \overline{\mu} g''(\overline{x})(\overline{\xi}, \overline{\xi}),$$

whence the conclusion. ▯

Remark 6. Under the assumptions of the lemma, $\chi(x, \mu) := \det(\mu I_n - g'(x))$ has root $\mu = 0$ for $x = \overline{x}$ with algebraic multiplicity $k \leq n$ (while the geometric multiplicity of 0, as an eigenvalue of $g'(\overline{x})$, is one). By the Malgrange preparation theorem [25], there are smooth scalar functions $a_0, \dots, a_{k-1}$ and b such that in the neighborhood of $(\overline{x}, 0)$,

$$\chi(x, \mu) = b(x, \mu)(\mu^k + a_{k-1}(x)\mu^{k-1} + \cdots + a_0(x))$$

and $a_0(\overline{x}) = \cdots = a_{k-1}(\overline{x}) = 0$, $b(\overline{x}, 0) \neq 0$. Accordingly,

$$(\det g')'(\overline{x}) = b(\overline{x}, 0)a_0'(\overline{x}).$$

The nondegeneracy at $\overline{x}$ is then equivalent to the statement that $\overline{x}$ is not a critical point of a_0, plus that g and a_0 are transverse at $\overline{x}$. The quantity a_0 can be interpretated as a smooth (and signed) singular value of g' when x is varied in the neighborhood of $\overline{x}$, as is clear from the following example. Take $g(x_1, x_2) = (x_2, x_1^2/2)$; $\overline{x} = (0, 0)$ is a nondegenerate corank one critical point. In a small enough neighborhood of $\overline{x}$, the smallest singular value of

$$g'(x) = \begin{bmatrix} 0 & 1 \\ x_1 & 0 \end{bmatrix}$$

is $\sigma(x) = |x_1|$, which is not differentiable at the critical point. In contrast, $a_0(x) = -x_1$ is smooth and provides the information needed to check nondegeneracy.

Proof of Theorem 4. Define the extended homotopy

$$\widetilde{h}(p_0, \lambda, t_c) = (x(t_f, x_0, p_0, \lambda), \det \frac{\partial x}{\partial p_0}(t_c, x_0, p_0, \lambda)).$$

By assumption, the point $(p_0(\overline{s}), \lambda(\overline{s}), t_f)$ belongs to $\{\widetilde{h} = 0\}$ and is regular. Indeed, vectors $(\delta p_0, \delta\lambda, \delta t_c)$ in the kernel of $\widetilde{h}'$ at this point verify $\delta\lambda = 0$ and

$$\frac{\partial h}{\partial p_0}(p_0(\overline{s}), \lambda(\overline{s})) \cdot \delta p_0 = 0,$$

$$\frac{\partial}{\partial p_0} \det \frac{\partial x}{\partial p_0}(t_f, x_0, p_0(\overline{s}), \lambda(\overline{s})) \cdot \delta p_0 + \frac{\partial}{\partial t_c} \det \frac{\partial x}{\partial p_0}(t_f, x_0, p_0(\overline{s}), \lambda(\overline{s})) \cdot \delta t_c = 0.$$

One has two cases depending on whether the partial derivative with respect to t_c vanishes in the second equation. If it does not, the kernel is one-dimensional and parameterized by

$$\delta p_0 \in \operatorname{Ker} \frac{\partial h}{\partial p_0}(p_0(\overline{s}), \lambda(\overline{s})).$$

If it does, δp_0 has to be zero since otherwise the previous lemmas would imply that

$$\frac{\partial}{\partial p_0} \det \frac{\partial x}{\partial p_0}(t_f, x_0, p_0(\overline{s}), \lambda(\overline{s})) \cdot \delta p_0 \neq 0$$

because of the order one assumption on the turning point. So the kernel of $\widetilde{h}'$ is parameterized by $\delta t_c \in \mathbf{R}$ and is also of dimension one. The extended homotopy is therefore well defined and regular in a neighborhood of $(p_0(\overline{s}), \lambda(\overline{s}), t_f)$. Parameterizing by arc length σ on $\{\widetilde{h} = 0\}$, one has $t_c(\overline{\sigma}) = t_f$. Since the point is a first turning point, it is enough to prove that $t_c'(\overline{\sigma}) \neq 0$ (here $' = d/d\sigma$); then necessarily $t_c'(\overline{\sigma}) < 0$, as there would otherwise be conjugate times on $(0, t_f)$ for $\sigma < \overline{\sigma}$ in $\{\widetilde{h} = 0\}$, that is, for $s < \overline{s}$ in $\{h = 0\}$. According to the description of the kernel,

$$\frac{\partial}{\partial p_0} \det \frac{\partial x}{\partial p_0}(t_f, x_0, p_0(\overline{\sigma}), \lambda(\overline{\sigma})) \cdot p_0'(\overline{\sigma}) + \frac{\partial}{\partial t_c} \det \frac{\partial x}{\partial p_0}(t_f, x_0, p_0(\overline{\sigma}), \lambda(\overline{\sigma})) \cdot t_c'(\overline{\sigma}) = 0.$$

The two previous alternatives result either in $p_0'(\overline{\sigma})$ being nonzero, in which case neither the first term in the sum nor $t_c'(\overline{\sigma})$ can vanish, or in $p_0'(\overline{\sigma}) = 0$, so $|t_c'(\overline{\sigma})| = 1$ (unit tangent vector). In both situations we conclude that $t_c'(\overline{\sigma})$ cannot be zero. ∎

In addition to turning points, there are two other issues on homotopy. First, when the connected component of the path considered is diffeomorphic to $\mathbf{R}$, boundary points (if any) are critical points of h. The classification of points in $\partial\Omega$ starts with the following result which is a simple consequence of the Morse lemma.

PROPOSITION 4 (see [2]). *Let $\overline{c} \in \{h = 0\}$ be a nondegenerate hyperbolic corank one critical point of h. Then, there are coordinates $d_1, \ldots, d_{n+1}$ such that in the neighborhood of $\overline{c}$, $\{h = 0\}$ is equal to*

$$d_1^2 - d_2^2 = 0, \quad d_3 = \cdots = d_{n+1} = 0.$$

In this case, we have a critical point jointly in (p_0, λ) and the intrinsic second order derivative is, up to a scalar,

$$\overline{\mu} h''(\overline{c})_{|\operatorname{Ker} h'(\overline{c}) \times \operatorname{Ker} h'(\overline{c})} \in \operatorname{Sym}(2, \mathbf{R}) \subset \mathbf{M}(2, \mathbf{R}) \simeq \mathscr{L}_2(\operatorname{Ker} h'(\overline{c}), \operatorname{Ker} h'(\overline{c}); \mathbf{R}),$$

where $\overline{\mu} \in (\mathbf{R}^n)^*$ is any nonzero covector with kernel $\operatorname{Im} h'(\overline{c})$. Hyperbolicity means that this order 2 symmetric matrix is nondegenerate with eigenvalues of opposite signs. As a consequence, the path of zeros is locally made of two smooth curves intersecting transversally, resulting in a bifurcation. The last issue is due to global features in parametric optimal control. For a given value λ_0 of the parameter, one has to compare the costs associated to zeros in each connected component of $\{h = 0\} \cap \{\lambda = \lambda_0\}$. Each zero of the shooting homotopy function defines which extremal and global solutions, if any, are those giving the infimum of the cost among them. In the three-body problem, the topology of the state manifold, $X_\mu = T^*Q_\mu$, comes into play; Q_μ has the topology of the eight curve with $\pi_1(Q_\mu) = \mathbf{Z} * \mathbf{Z}$, and a heuristic classification of extremals based on homology is proposed in [14].

Two homotopies are used to compute numerically minimum time trajectories of the restricted three-body problem. A continuation on the ratio of masses, μ, is first considered. In practice, the isolated contacts with the codimension two switching surface are neglected, and we restrict the computation to smooth extremals without π-singularities. This yields regularity of the μ-parameterized minimum time problem in view of the following.

LEMMA 7. *In the absence of π-singularities, the Legendre condition holds.*

Proof. If the switching function never vanishes, one has $|u| = 1$ everywhere. So we can restrict the control set U to $\mathbf{S}^1$. The manifold is without boundary, and in the chart $u = (\cos\alpha, \sin\alpha)$, $\alpha \in \mathbf{R}$, one has[10]

$$H(x, \alpha, p) = p^0 + H_0(x, p) + \varepsilon(\cos\alpha H_1(x, p) + \sin\alpha H_2(x, p)).$$

Accordingly,

$$\nabla^2_{\alpha\alpha} H(x, \alpha, p) = -\varepsilon(\cos\alpha H_1(x, p) + \sin\alpha H_2(x, p)).$$

Along an extremal, $\nabla^2_{\alpha\alpha} H(x(t), \alpha(t), p(t)) = -\varepsilon|\psi(t)|$ which is bounded over by some negative constant on $[0, t_f]$ as ψ is smooth and nonvanishing. ∎

Using previous knowledge on the two-body minimum time trajectories [15, 16], we are able to compute transfers from a circular orbit around the first primary toward the L_2 point when $\mu = 0$ and then to follow the path until any value $\mu \in (0, 1)$. The absence of conjugate points—ensuring local optimality—is checked along the path using the `hampath` code [12] that embeds the relevant rank test; see Figure 4. A continuation on the target eventually allows one to obtain solutions, for instance, in the earth-moon system ($\mu \simeq 1.21e-2$) from a geostationary orbit to a circular lunar one for average values of the control magnitude ε; see Figure 5. To reach lower values of ε, a continuation on this parameter is finally employed as in [15], which emphasizes the role of the topology of the state space previously mentioned. Many local minima exist, yielding as many zeros of the shooting function. When decreasing ε, at some point on the resulting path the number of revolutions around the first primary has to be increased to retain global optimality, which means using a heuristic to jump to another connected component (*branch*) of the zero set (see Figure 6). Practically, this is done here by augmenting heuristically the initial guess for the final time to force shooting to converge to a solution with a larger number of revolutions; using this value as a starting point, a part of the new branch can then be computed by differential continuation. The situation is analogous to the one in Riemannian geometry with cut and conjugate points: Up to some point, the path provides minimizers; then global optimality is lost (typically because of the topology of the manifold), but local optimality persists up to another point. Past this second point (a turning point, in the simple case we framed in the beginning of the section), even local optimality is lost (see Figure 7). Table 1 summarizes the results obtained.

The computations combining shooting and homotopy presented here are meant to initialize the solution of more complicated problems. A time minimum trajectory of the three-dimensional model of the SMART-1 mission [31] is given Figure 8. The three-dimensional case proves to be much more difficult to solve numerically than the coplanar one; this is probably due to the angle between the planes containing the initial and final orbit. Multiple shooting provided by `hampath` and initialized by results on the coplanar model is used; two arcs are considered with a junction point in

[10]The Hamiltonian lift H_0 implicitly depends on μ, since $F_0 = \overrightarrow{J_\mu}$ (compare section 1).

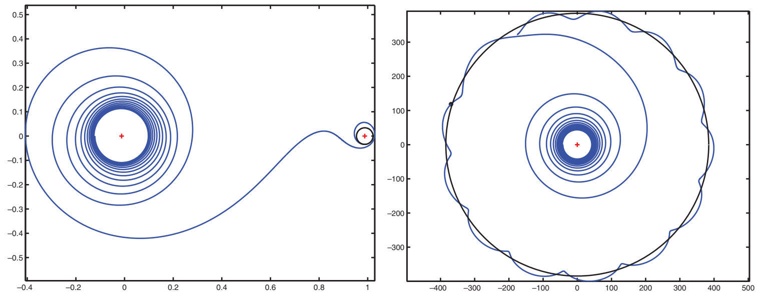

FIG. 5. *Minimum time trajectory in the earth-moon system* ($\mu \simeq 1.21e-2$, $\varepsilon = 2.44e-1$). *Left, in the rotating frame; right, in the fixed frame to emphasize capture by the second primary at the end of the transfer. Before the capture, the trajectory approaches the projection of the* L_2 *point in the* (q_1, q_2)*-plane.*

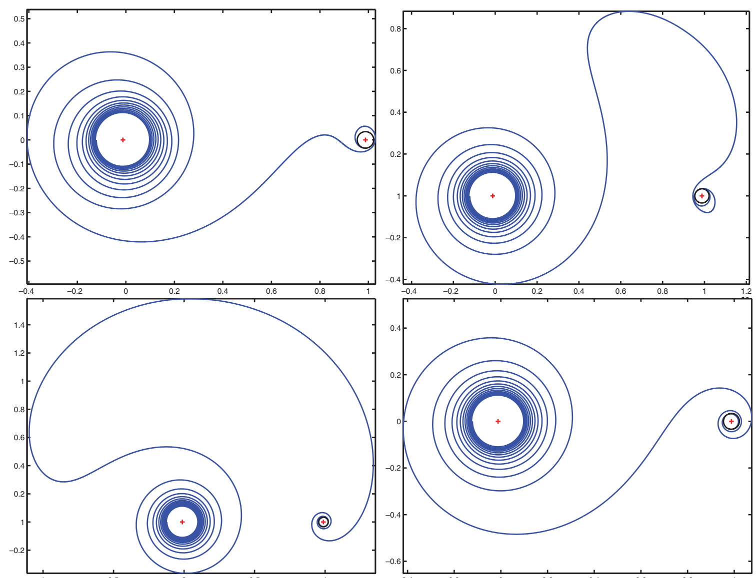

FIG. 6. *Minimum time trajectory for* ε *between* $2.44e-1$ *and* $2.196e-1$ ($\mu \simeq 1.21e-2$, *rotating frame). As the control magnitude is decreased, strategies are evolved. In the upper graphs, the first two extremals have the same rotation number around the first primary and both wind around the second one positively. Conversely, the third extremal (bottom left) winds negatively around the second primary, while the fourth (bottom right) makes an additional revolution around the first one.*

the neighborhood of the L_2 Lagrange point. Work in progress includes the treatment of the maximization of the final mass.

Conclusion. In this paper, we have given a controllability result for the restricted three-body problem; under mild assumptions, two orbits around the primaries with Jacobian constants less than the Jacobian at the L_1 Lagrange point can be connected. Using the control-affine structure of the dynamics, we have given a primary classification of extremals and provided global bounds on the number of switchings of time minimizing controls. Homotopy techniques are instrumental in solving numerically the problem which has natural small parameters; the link between turning points and local optimality of extremals along the path of zeros of a parameterized shooting

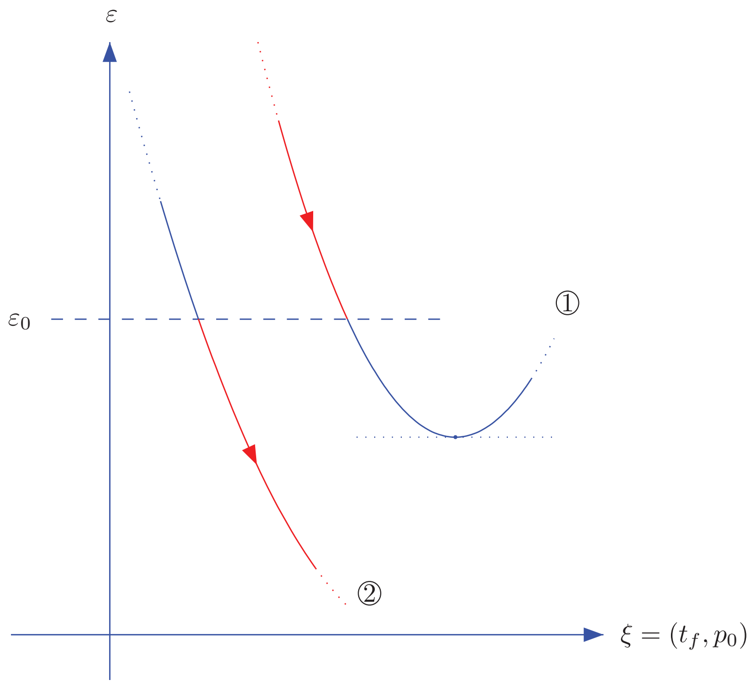

FIG. 7. *Continuation on ε. Down to ε_0, branch* 1 *yields minimizers. In the case of free final time, the shooting unknown is $\xi = (t_f, p_0)$, p_0 belonging to the zero level set of the Hamiltonian. Past ε_0, global optimality is lost on branch* 2 *and a switch to branch* 2 *has to be made (resulting in a loss of regularity of the value function). Past the turning point on branch* 1 *(conjugacy of the target point), even local optimality is lost.*

TABLE 1
Earth-moon system ($\mu \simeq 1.21e-2$). Minimum time t_f from the geostationary orbit to the L_2 Lagrange point, and first conjugate time t_{1c}. That $t_{1c} > t_f$ ensures local optimality of the computed extremal.

ε	t_f	t_{1c}
2.4405	1.4705	2.2750
0.2440	8.4401	10.640
0.2221	9.7710	12.045
0.2026	11.152	13.500
0.1806	13.157	15.595
0.1586	14.369	16.900
0.1293	18.024	20.700
0.1074	21.323	24.125
0.0732	32.216	35.295
0.0437	51.504	54.930

function has been established in a simple framework. This preliminary analysis of the problem has allowed us to compute minimum time solutions for the boundary conditions of the SMART-1 mission using a two to three-body continuation.

Future work could be devoted to reaching very low thrusts, typical of this kind of mission. The performance index should also be changed to consider instead minimization of the fuel consumption, equivalent to minimizing the L^1-norm of the control. The final time should then be fixed, $t_f = c \cdot \overline{t}_f$, $c \geq 1$, where $\overline{t}_f$ is the minimum time for the prescribed boundary conditions. The additional difficulty in this prob-

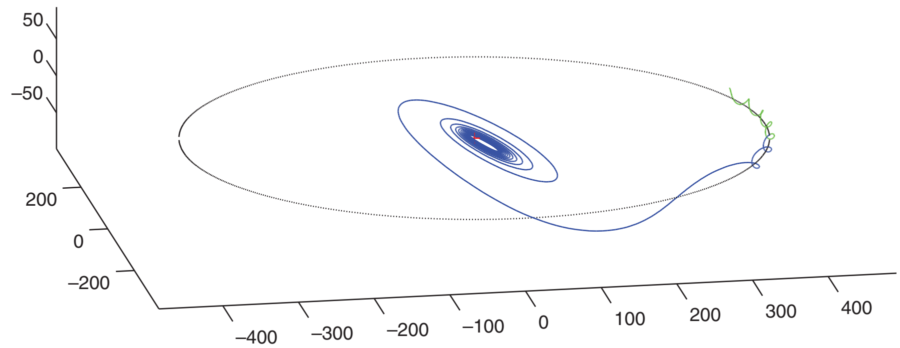

FIG. 8. *Three-dimensional minimum time transfer, SMART-1 boundary conditions (fixed frame). The control magnitude ε is 0.7 Newtons for an initial mass of 350 kilograms and a specific impulse of 1640 seconds. (The variation of mass has been taken into account for this simulation; see [31]). The final time is 26.2 days. The dotted black circle represents the orbit of the moon. The green trajectory represents the uncontrolled motion after capture by the moon.*

lem comes from the existence for $c > 1$ of ballistic arcs $u = 0$ (zero-bang structure of $|u|$). A related issue would be to use a four-body model so as to include the existence of "cheap" trajectories [26] whose existence relies on the presence of the fourth body.

Acknowledgments. The authors thank Joseph Gergaud for introducing them to homotopy in optimal control and Bernard Bonnard for his geometric influence on this work.

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