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ON SINGULARITIES OF MINIMUM TIME CONTROL-AFFINE SYSTEMS*

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Abstract. Affine control problems arise naturally from controlled mechanical systems. Building on previous results [Agrachev and Biolo, *J. Dyn. Control Syst.*, 23 (2017), pp. 577–595; Caillau and Daoud, *SIAM J. Control Optim.*, 50 (2012), pp. 3178–3202], we prove that, in the case of time minimization with control on the disk, the extremal flow given by Pontrjagin's maximum principle is smooth along the strata of a well-chosen stratification. We also study this flow in terms of regular-singular transition and prove that the singularity along time-minimizing extremals crossing these strata is at most logarithmic. We then apply these results to mechanical systems, paying special attention to the case of the controlled three-body problem.

Key words. minimum time, normal hyperbolicity, regular-singular transition, control switching

AMS subject classifications. 49K15, 49N60, 37D20, 70Q05

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1. Introduction.

The aim of this paper is the study of time minimization for control-affine systems of the form

$$\dot{x}(t) = F_0(x(t)) + u_1(t)F_1(x(t)) + u_2(t)F_2(x(t)), \quad u_1^2(t) + u_2^2(t) \leq 1,$$

where the control $u = (u_1, u_2)$ is thus contained in the disk and where all vector fields are smooth. For the sake of simplicity, we carry out the reasoning in a 4-dimensional manifold—which is the most relevant for the applications—but the method and results of section 3 can be adapted to a $2m$-dimensional manifold with an m-dimensional control. The reason for studying the singularities of such optimal control systems is that the regularity of the Hamiltonian flow given by Pontrjagin's maximum principle—the *extremal flow*—is crucial to study necessary conditions for optimality and thus to determine the actual optimal solutions. The numerical study of these problems also strongly depends on regularity. Recently, new genericity results for these kinds of singularities have been obtained in [6] for control-affine systems of any dimension with an even number of controls prescribed to the closed unit ball. A similar analysis has been conducted for control constrained in a polyhedron in [19]. (See the beginning of section 3 for further comments.) Sufficient conditions for optimality have also been proved in [3] for a minimum time control-affine system in a related context.

In the first section, we recall for the sake of completeness some classical results of geometric optimal control, with an emphasis on the Pontrjagin maximum principle, which is key to our study and reduces the problem to the study of a singular

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Hamiltonian system. The singularities of this system are related to the discontinuities of the optimal control u, often called *switches*. We study the local structure of the Hamiltonian flow under generic assumptions in section 3. The beginning of our study builds on the analysis in [9] and goes one step further than the recent paper [1], where the flow is proved to be welldefined and continuous (see also [2] in higher dimensions): Using the underlying normal hyperbolicity of the system, we provide a stratification whose strata carry a smooth flow; the extremal flow is thus piecewise smooth and continuous. This regular behavior is obtained as a consequence of a straightening normal form lemma, around the singular locus, proved at the end of the section. In section 4, we investigate the kind of singularity of the flow encountered when crossing strata. Thanks to a suitable normal form, we prove that the associated regular-singular transition results in a logarithmic term, implying that the flow belongs to the $\log - \exp$ category [24]. This behavior of the flow strongly contrasts with the situation studied in [11], where singularities of the flow were stable; here, singularities are destroyed by small perturbations of the initial conditions, unless these perturbations belong to a codimension one stratum. We then apply these results to the controlled circular restricted three-body problem in section 5. We finally investigate global properties of the flow and give upper bounds on the number of switches of the control for this particular nonlinear system. Note that such bounds for time minimization are given in the linear case in [5]. In contrast to [10], where a subset of the switching set was studied, we treat here the general case using a comparison principle.

2. Setting. Let M be a smooth (that is $\mathscr{C}^\infty$-smooth) 4-dimensional manifold, and let us consider the following control-affine system:

(2.1)
$$\dot{x}(t) = F_0(x(t)) + u_1(t)F_1(x(t)) + u_2(t)F_2(x(t)), \quad |u(t)| = \sqrt{u_1^2(t) + u_2^2(t)} \leq 1.$$

Given endpoint conditions $x(0) = x_0$, $x(t_f) = x_f$, one can consider the minimization of the final time, t_f. The corresponding Hamiltonian writes

$$H(x, p, u) = H_0 + u_1\, H_1 + u_2\, H_2, \quad H_i := \langle p, F_i(x)\rangle, \quad i = 0, 1, 2,$$

and the classical Pontrjagin maximum principle [4] provides a necessary condition for optimality, allowing us to work with a true—that is, independent of the control—Hamiltonian system yet at the expense of introducing singularities.

THEOREM 2.1 (Pontrjagin maximum principle). *Let $u : [0, t_f] \to \mathbf{R}^2$ be an essentially bounded time minimizing control of* (2.1)*, and let x be the associated trajectory. There exists a Lipschitz curve $p(t) \in T^*_{x(t)}M$ such that, almost everywhere on* $[0, t_f]$*,*

$$\dot{x}(t) = \frac{\partial H}{\partial p}(x(t), p(t), u(t)), \quad \dot{p}(t) = -\frac{\partial H}{\partial x}(x(t), p(t), u(t)), \tag{2.2}$$

$$H(x(t), p(t), u(t)) = \max_{|v| \leq 1} H(x(t), p(t), v), \tag{2.3}$$

and $H(x(t), p(t), u(t)) \geq 0$. Moreover, p does not vanish on $[0, t_f]$.

Triple (x, p, u) solutions of (2.2)–(2.3) are called *extremals* and their projections on M *extremal trajectories*. We denote by $z = (x, p)$ elements of the cotangent bundle T^*M, where p belongs to the fiber T^*_xM. We define the *switching surface*:

$$\Sigma := \{z = (x, p) \in T^*M \mid H_1(z) = H_2(z) = 0\}.$$

Extremals along which H_1 and H_2 do not vanish simultaneously are called *bang arcs*. An extremal is said to be *bang-bang* if it is a concatenation of bang arcs. The following proposition is clear.

PROPOSITION 2.2. *An extremal lying out of Σ is an integral curve of the maximized Hamiltonian*

$$H_0(z) + \sqrt{H_1^2(z) + H_2^2(z)}.$$

The associated control belongs to $\mathbf{S}^1$ and is equal to

$$u = \frac{1}{\sqrt{H_1^2 + H_2^2}}(H_1, H_2). \tag{2.4}$$

Let now $\overline{z} = (\overline{x}, \overline{p})$ belong to Σ. We are interested in the local behavior of the extremal flow in a neighborhood of this singular point. We make the following transversality assumption (remember that the ambient manifold is 4-dimensional):

$$\mathrm{Span}_{\overline{x}}\{F_1, F_2, F_{01}, F_{02}\} = T_{\overline{x}}M, \tag{A}$$

where $F_{ij} := [F_i, F_j]$ denotes the Lie bracket of vector fields. As a derivation on functions, $[F_i, F_j]$ is the commutator $F_iF_j - F_jF_i$, while in coordinates $[F_i, F_j](x) = F_j'(x)F_i(x) - F_i'(x)F_j(x)$. Property (A) is generic among vector fields and points of Σ and holds in particular for control systems arising from mechanical systems (see section 5). Since the adjoint vector cannot be zero, assumption (A) implies that, for z in a neighborhood of $\overline{z}$,

$$H_1^2(z) + H_2^2(z) + H_{01}^2(z) + H_{02}^2(z) \neq 0,$$

where $H_{ij} := \{H_i, H_j\}$ now denotes the Poisson bracket of functions on the cotangent bundle. In accordance with the definition of Lie brackets, in coordinates,

$$\{H_i, H_j\} = \sum_{k=1}^{n} \frac{\partial H_i}{\partial p_k}\frac{\partial H_j}{\partial x_k} - \frac{\partial H_i}{\partial x_k}\frac{\partial H_j}{\partial p_k}.$$

3. Stratification of the extremal flow.

Following [9], we partition Σ according to

$$\Sigma_- := \{z \in \Sigma \mid H_{12}^2(z) < H_{02}^2(z) + H_{01}^2(z)\},$$
$$\Sigma_+ := \{z \in \Sigma \mid H_{12}^2(z) > H_{02}^2(z) + H_{01}^2(z)\},$$
$$\Sigma_0 := \{z \in \Sigma \mid H_{12}^2(z) = H_{02}^2(z) + H_{01}^2(z)\}.$$

In this paper, we focus on the case Σ_-, which is the relevant one for mechanical systems, as explained section 5. The other situations (in particular Σ_0) will be tackled in a forthcoming work [17]. The main results of this section refine the analysis in [1] by using normal hyperbolicity of the system to provide a suitable stratification under our standing assumption (A).

PROPOSITION 3.1. *Let $\overline{z}$ belong to Σ_-; there exists a neighborhood $O_{\overline{z}}$ of $\overline{z}$ in T^*M such that* (i) *for every $z \in O_{\overline{z}}$ there exists a unique extremal passing through z;* (ii) *every such extremal intersects Σ at most once in $O_{\overline{z}}$.*

Remark 3.2. In the terminology of [6], Σ_- points along the extremal are called *Fuller times of order zero*. In that paper, *Fuller times of higher order* are defined inductively, and it is shown that, generically, there are only finite-order Fuller times.

An upper bound on the order exists that only depends on the dimension of the ambient manifold. We focus here on the structure of the extremal flow in the neighborhood of a single Σ_- point. As we shall see in section 5, for usual mechanical systems, $\Sigma = \Sigma_-$; this precludes accumulation of switching points (Fuller phenomenon).

Let us consider an extremal $z : [0, t_f] \to T^*M$ as in Proposition 3.1, initializing at $z(0) = \overline{z}_0$ and crossing Σ once at $\overline{z} \in \Sigma_-$ and time $\overline{t} \in (0, t_f)$. The following holds.

THEOREM 3.3. *There exists an open neighborhood $O_{\overline{z}_0} \subset T^*M \backslash \Sigma$ of $\overline{z}_0$ such that the extremal flow $(t, z_0) \mapsto z(t, z_0)$ is well defined and continuous on $[0, t_f] \times O_{\overline{z}_0}$. Moreover, $O_{\overline{z}_0} = S^0 \cup S^s$, where S^s is a codimension one submanifold of initial conditions leading to Σ_- and where $S^0 = O_{\overline{z}_0} \backslash S^s$. The time $t_\Sigma(z_0)$ to reach Σ from an initial condition z_0 is well defined and smooth for z_0 in S^s. Both S^0 and S^s are stable by the flow, which is smooth on $[0, t_f] \times S^0$ and on $([0, t_f] \times S^s) \backslash \Delta$, where $\Delta = \{(t_\Sigma(z_0), z_0),\ z_0 \in S^s\}$.*

Remark 3.4. Although the analysis is drawn on a 4-dimensional manifold with control on the 2-disk, it will be clear from the proofs that the same results hold in dimension $2m$ with control on the m-dimensional unit ball. In addition, while we assume that the reference extremal departing at $\overline{z}_0$ crosses only once Σ for $t \in [0, t_f]$, the analysis can be readily extended to an extremal with a finite number of contacts with Σ at Σ_- points.

Let us provide a simple example to illustrate the situation described by Proposition 3.1 and Theorem 3.3. Consider the control system

$$\left\{ \begin{array}{ll} \dot{x}_1(t) = 1 + x_3(t), & \dot{x}_3(t) = u_1(t), \\ \dot{x}_2(t) = x_4(t), & \dot{x}_4(t) = u_2(t), \end{array} \right. \tag{3.1}$$

with control in the 2-disk, $u_1^2 + u_2^2 \leq 1$. The maximized Hamiltonian (from the Pontrjagin maximum principle) is

$$H(x, p) = p_1(1 + x_3) + p_2 x_4 + \sqrt{p_3^2 + p_4^2}$$

(p_i being the adjoint variable of x_i), and the codimension two submanifold

$$\Sigma = \{p_3 = 0\} \cap \{p_4 = 0\} = \Sigma_-$$

is the singular locus. (Note that the distribution $\{F_1, F_2\}$ is involutive, so the bracket H_{12} vanishes.) The adjoint states p_1 and p_2 are constant; let $a = -p_1(0)$ and $c = -p_2(0)$. We get $p_3(t) = at + b$, $p_4(t) = ct + d$, with $b = p_3(0)$ and $d = p_4(0)$. Then

$$\dot{x}_3(t) = \frac{at + b}{\sqrt{(at + b)^2 + (ct + d)^2}}\,, \tag{3.2}$$

so that singularities occur when $(at + b, ct + d)$ vanishes for some t, that is, when $ad - bc = 0$, which defines the codimension one submanifold $S^s = \{p_1 p_4 - p_2 p_3 = 0\} \setminus \{p = 0\}$ (remember that the adjoint state p cannot be zero for a minimumtime extremal by virtue of the maximum principle). We get symmetric dynamics for x_4 and end up with the same submanifold. One verifies that this stratum is stable by the flow of the maximized Hamiltonian. Outside S^s, we can explicitly solve (3.2) and obtain

$$\begin{aligned} x_3(t, z_0) = x_3(0) &+ \frac{a}{a^2 + c^2}\left(\sqrt{(at + b)^2 + (ct + d)^2} - \sqrt{b^2 + d^2}\right) \\ &-c\frac{ad - bc}{(a^2 + c^2)^{3/2}}\left[\operatorname{argsh}\left(\frac{(a^2 + c^2)t + ab + cd}{ad - bc}\right) - \operatorname{argsh}\left(\frac{ab + cd}{ad - bc}\right)\right]. \end{aligned}$$

It is clear that the flow is smooth outside S^s. If a and c are zero, p_3 and p_4 become constant, and since p cannot vanish, there are no contacts with Σ. Now observe that the flow is defined on S^s by

$$x_3(t, z_0) = x_3(0) + \frac{a}{a^2 + c^2}(\sqrt{(at+b)^2 + (ct+d)^2} - \sqrt{b^2 + d^2})$$

for all $z_0 \in S^s \setminus \{p = 0\}$. Restricted to S^s, the flow is smooth outside switches. We also have global continuity on T^*M but not Lipschitz continuity. Furthermore, on this simple model, a singularity of the $y \ln y$ type appears when crossing S^s, that is, when the determinant $ad-bc$ goes to zero. We prove in section 4 that the singularities in the general case are not worse than in this example.

3.1. Proof of Proposition 3.1. According to assumption (A), the mapping

$$(x, p_1, p_2, p_3, p_4) \mapsto (x, H_1, H_2, H_{01}, H_{02})$$

defines a smooth change of coordinates in a small enough neighborhood $O_{\overline{z}}$ of $\overline{z}$. A polar blowup is used to study the dynamics near the singularity, adding an $\mathbf{S}^1$-fiber above $\rho = 0$:

$$(H_1, H_2) = (\rho\cos\theta, \rho\sin\theta), \quad (\rho, \theta) \in \mathbf{R} \times \mathbf{S}^1.$$

In polar coordinates, (2.4) reads $u = (\cos\theta, \sin\theta)$ and $\Sigma = \{\rho = 0\}$. Computing, the dynamics are

$$X : \begin{cases} x' = \rho(F_0(x) + \cos\theta \cdot F_1(x) + \sin\theta \cdot F_2(x)), \\ \rho' = \rho(\cos\theta \cdot H_{01} + \sin\theta \cdot H_{02}), \\ \theta' = H_{12} + \cos\theta \cdot H_{02} - \sin\theta \cdot H_{01}, \\ H'_{01} = \rho(H_{001} + \cos\theta \cdot H_{101} + \sin\theta \cdot H_{201}), \\ H'_{02} = \rho(H_{002} + \cos\theta \cdot H_{102} + \sin\theta \cdot H_{202}), \end{cases} \tag{3.3}$$

where we have changed time from t to s ($' = \mathrm{d}/\mathrm{d}s$) with $\mathrm{d}t = \rho\,\mathrm{d}s$ to get rid of the $1/\rho$ singularity on $\dot\theta$ and denoted X the corresponding vector field. In this new time, the autonomous vector field in the right-hand side of (3.3) is smooth, which implies existence and uniqueness of maximal solutions through a point, as well as smoothness of the flow. Note that when ρ vanishes, only θ is not constant; in particular, $\Sigma = \{\rho = 0\}$ is invariant by the flow. In the following, we denote $\overline{H}_{ij} = H_{ij}(\overline{z})$, $i, j = 0, 1, 2$. The next lemma establishes that in each part of Σ, the derivative of θ has a different number of equilibria.

LEMMA 3.5. *Let z belong to Σ; the mapping $\theta \mapsto H_{12} + \cos\theta \cdot H_{02} - \sin\theta \cdot H_{01}$ has* (i) *two zeros, denoted by θ_- and θ_+, for z in Σ_-;* (ii) *exactly one zero for z in Σ_0;* (iii) *no zero for z in Σ_+. In the $z \in \Sigma_-$ case, the two mappings $(x, H_{01}, H_{02}) \mapsto \theta_\pm$ are well defined and smooth in a neighborhood of $(\overline{x}, \overline{H}_{01}, \overline{H}_{02})$.*

Proof. Setting $(H_{01}, H_{02}) = (r\cos\phi, r\sin\phi)$, where $r \neq 0$ under assumption (A),

$$H_{12} + \cos\theta \cdot H_{02} - \sin\theta \cdot H_{01} = H_{12} - r\sin(\theta - \phi),$$

so $H_{12}/r = \sin(\theta - \phi)$ has two solutions, θ_- and θ_+, if $\overline{z} \in \Sigma_-$; no solution if $z \in \Sigma_+$; and exactly one if $z \in \Sigma_0$ (since $H_{12}(z)/r = \pm 1$). The variables $(x, \theta, H_{01}, H_{02})$ define a local chart on $\Sigma = \{\rho = 0\}$, and the mapping

$$g : (x, \theta, H_{01}, H_{02}) \mapsto H_{12} + \cos\theta \cdot H_{02} - \sin\theta \cdot H_{01}$$

verifies

$$\frac{\partial g}{\partial \theta}(x, \theta_\pm, H_{01}, H_{02}) = -r\cos(\theta_\pm - \phi) = \pm\sqrt{r^2 - H_{12}^2} \neq 0,$$

so the implicit function theorem allows to conclude. ∎

To complete the proof of Proposition 3.1, let us recall that a diffeomorphism f of manifold M onto itself is said to be normally hyperbolic along a compact submanifold N if the submanifold N is invariant by f and if (i) every fiber of the tangent bundle of M along N admits a splitting $T_xM = E^u(x) \oplus T_xN \oplus E^s(x)$ for all $x \in N$ such that $f'(x) \cdot E^s(x) = E^s(f(x))$ and $f'(x) \cdot E^u(x) = E^u(f(x))$ (f preserves the splitting); (ii) there exists $\lambda_1 \leq \mu_1 < \lambda_2 \leq \mu_2 < \lambda_3 \leq \mu_3$, with $\mu_1 < 1 < \lambda_3$, such that (the endomorphism norm below being induced by a given Riemannian structure on M)

$$\lambda_1 \leq |f'_{|E^s}| \leq \mu_1, \quad \lambda_2 \leq |f'_{|TN}| \leq \mu_2, \quad \lambda_3 \leq |f'_{|E^u}| \leq \mu_3. \tag{3.4}$$

The distributions E^s and E^u are locally integrable, and one can Construct the local stable and unstable manifolds, $W(x)^s$ and $W(x)^u$, tangent respectively, to $E^s(x)$ and $E^u(x)$ at each point $z \in N$. One also defines $\mathscr{W}^s := \bigcup_{x\in N} W^s(x)$ and $\mathscr{W}^u := \bigcup_{x\in N} W^u(x)$, the local stable and unstable manifolds of N. Let also l_s and l_u be the biggest integers such that $\mu_1 \leq \lambda_2^{l_u}$ and $\mu_2^{l_s} \leq \lambda_3$. The following holds (see (Theorem 3.5 in [14] or [18]), giving the regularity of the stable and unstable manifolds in terms of the ratio of the contraction and expansion rates.

THEOREM 3.6 (Hirsch, Pugh, Shub). *Any f-invariant submanifold which is close enough to N is included in $\mathscr{W}^s \cup \mathscr{W}^u$. Furthermore, $\mathscr{W}^s$ and $\mathscr{W}^u$ are submanifolds of class $\mathscr{C}^{l_s}$ and $\mathscr{C}^{l_u}$, respectively.*

In our case, this result is applied on the cotangent bundle T^*M of the original state manifold M, and we have two codimension two submanifolds of equilibrium points, namely (Figure 1),

$$z_\pm = (x, \rho = 0, \theta = \theta_\pm(x, H_{01}, H_{02}), H_{01}, H_{02}),$$

parameterized locally by $y := (x, H_{01}, H_{02})$ in a neighborhood of $(\overline{x}, \overline{H}_{01}, \overline{H}_{02})$. We set

$$\cos\theta_- \cdot H_{01} + \sin\theta_- \cdot H_{02} = -\sqrt{r^2 - H_{12}^2} < 0 \tag{3.5}$$

and the opposite for θ_+. The Jacobian of the system (3.3) has two nonzero eigenvalues at those points: $\cos\theta_\pm \cdot H_{01} + \sin\theta_\pm \cdot H_{02}$ and their opposite and a 6-dimensional kernel. So we have a 1-dimensional stable submanifold $W^s(z_\pm)$ and a 1-dimensional unstable submanifold $W^u(z_\pm)$ in every equilibrium $z_\pm$. The flow is thus normally hyperbolic to the manifold

$$N_- = \{z = z_-(y) = (\rho = 0, \theta = \theta_-(y), y)\}$$

and symmetrically to

$$N_+ = \{z = z_+(y) = (\rho = 0, \theta = \theta_+(y), y)\}.$$

We now focus on the case of N_-, the analysis being the same for N_+. On N_-, the dynamics are trivial: Every point is an equilibrium. Hence, there exists a unique

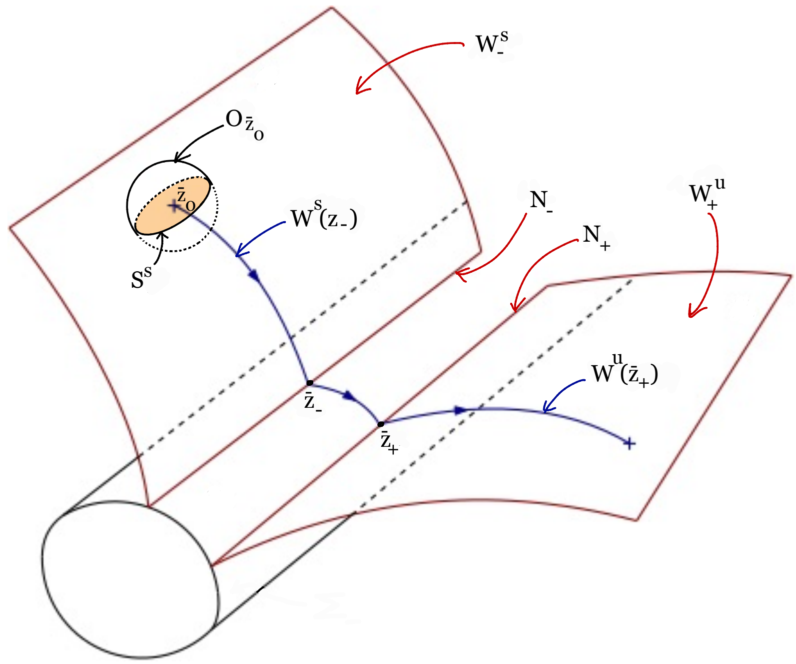

FIG. 1. *Switching set* Σ_- *and extremal flow after blowup and stratification.*

trajectory converging to z_- when $s \to \infty$ in the stable manifold $W^s(z_-)$. On Σ, everything is constant but θ, which realizes a heteroclinic connection from θ_- to θ_+. Symmetrically, there is one trajectory converging to z_+ when $s \to -\infty$ in the unstable manifold $W^u(z_+)$. Blowing down from (ρ, θ) to (H_1, H_2), there is a unique extremal passing through every $z \in \Sigma$ in a small enough neighborhood $O_{\overline{z}}$ of $\overline{z}$, and those extremals cross Σ only once if the neighborhood is small enough. Furthermore, in the original time t, this happens in finite time for any initial conditions in $O_{\overline{z}}$ leading to the singular locus. Indeed, note that the negative expression in (3.5) is smooth and bounded on $O_{\overline{z}}$. Let $C < 0$ be a negative upper bound; given the dynamics of ρ, one has $\rho'(s) \leq \rho(s)C$ and $\rho(s) \leq \rho(0)e^{Cs}$ by Gronwall's lemma. So the time t required to reach Σ is bounded by

$$\int_0^\infty \rho(s)\,\mathrm{d}s \leq -\frac{\rho(0)}{C} < \infty.$$

All in all, through every z in $O_{\overline{z}}$ passes a unique extremal either crossing Σ exactly once or not crossing Σ at all. This concludes the proof of Proposition 3.1.

3.2. Proof of Theorem 3.3. Let $z : [0, t_f] \to T^*M$ be an extremal departing from some $\overline{z}_0$ and crossing Σ at $\overline{z} \in \Sigma_-$ and $\overline{t} \in (0, t_f)$. According to what we have proved in the previous subsection, (2.2)–(2.3) admits a unique solution defined on $[0, t_f]$ for an initial condition z_0 close enough to $\overline{z}_0$. So there exists a small enough open neighborhood $O_{\overline{z}_0}$ of $\overline{z}_0$ such that the flow $z(t, z_0)$ of (2.2)–(2.3) is well defined for $(t, z_0) \in [0, t_f] \times O_{\overline{z}_0}$. In particular, $z(t, \overline{z}_0) = z(t)$, the reference extremal. We use the normally hyperbolic invariant manifold N_- previously constructed to define $\mathscr{W}^s_- := \bigcup_{z \in N_-} W^s(z)$. Since N_- is made of equilibria, $\lambda_2 = \mu_2 = 1$ in the splitting (3.1) at any point of N_-, and $\mathscr{W}^s_-$ is a $\mathscr{C}^\infty$-smooth submanifold whose dimension is

$7 = \dim N + 1$, every fiber $W^s(z)$ being of dimension one. The stratum we look for is

$$S^s := \mathscr{W}^s_- \cap \{\rho > 0\}.$$

One can similarly define $\mathscr{W}^u_+ := \bigcup_{z \in N_+} W^u(z)$, also $\mathscr{C}^\infty$-smooth of codimension one, and $S^u := \mathscr{W}^u_+ \cap \{\rho > 0\}$.

To understand the regularity of the flow on this strata, we use a normal form to rewrite the system in the neighborhood of the equilibria θ_-. (The same approach also works near θ_+.) Using again polar coordinates $(H_{01}, H_{02}) = (r\cos\phi, r\sin\phi)$, the dynamics (3.3) write

$$\begin{cases} \rho' = r\rho\cos(\theta - \phi), \\ \theta' = H_{12} - r\sin(\theta - \phi), \\ \xi' = \rho h(\rho, \theta, \xi), \end{cases} \tag{3.6}$$

where $\xi = (x, r, \phi)$ and h is a smooth function. We set $\psi = \theta - \phi$, rescale the time according to $\mathrm{d}v = r\,\mathrm{d}s$ (as (A) implies $r > 0$ in the neighborhood of $\overline{z}$), and study a system with the following structure (the derivation w.r.t. v still being noted $'$):

$$\begin{cases} \rho' = \rho\cos\psi, \\ \psi' = g(\rho, \psi, \xi) - \sin\psi =: G(\rho, \psi, \xi), \\ \xi' = \rho h(\rho, \psi, \xi), \end{cases} \tag{3.7}$$

where g is a smooth function (so is G) defined on an open set O of $\mathbf{R}^2 \times D$, D being a compact domain of $\mathbf{R}^6$. As H_{12} is a smooth function in $(H_1, H_2) = (\rho\cos\theta, \rho\sin\theta)$, because it is smooth w.r.t. the change of coordinates given by assumption (A),

$$g(\rho, \psi, \xi) = a(\xi) + \rho b(\xi, \psi) + O(\rho^2)$$

since when $\rho = 0$, g does not depend anymore on θ. In addtion, $|g| < 1$ on O since it is a small neighborhood of Σ_-. Equilibria occur when $\rho = G = 0$. They are semi-hyperbolic since they are outside $\{\psi = \pm\pi/2\}$ for $\overline{z}$ in Σ_-. More precisely, it was shown in the previous subsection that the flow of this system is normally hyperbolic to the manifold $\{\rho = 0\} \cap \{G = 0\}$. For each ξ, we retrieve the two equilibria $z_\pm$, and the previously defined $\theta_\pm$ are mapped to $\psi_\pm$ in the new set of coordinates for the blowup. Indeed, thanks to the structure of g, we get $\partial g/\partial\psi(0, \psi, \xi) = 0$, so

$$\frac{\partial G}{\partial \psi}(0, \psi, \xi) = -\cos\psi \neq 0,$$

and there exist two smooth functions $\psi_\pm(\xi)$ that allow to parameterize anew the two pieces of $\{G = 0\} \cap \{\rho = 0\}$ according to

$$N_\pm = \{z = z_\pm(\xi) = (\rho = 0, \psi = \psi_\pm(\xi), \xi)\}.$$

The following result is a preparation lemma for the normal form computation.

Lemma 3.7. *In a neighborhood of N_-, the vector field X giving system* (3.3) *is smoothly conjugated to a vector field Y (that is, there exists a smooth diffeomorphism Ψ s.t. $\Psi_* X = Y$) such that*

$$Y : \begin{cases} \rho' = -\rho(1 + O(|\rho| + |\omega|)), \\ \omega' = \omega + O((|\rho| + |\omega|)^2), \\ \xi' = \rho\, O(|\rho| + |\omega|) \end{cases} \tag{3.8}$$

in suitable coordinates (ρ, ω, ξ).

Proof. Let us set $\omega = \psi - \psi_-(\xi)$ along $\{G = 0\}$ and study the system near $\omega = 0$. In these new coordinates,

$$\begin{cases} \rho' = \rho\cos(\omega + \psi_-(\xi)), \\ \omega' = g(\rho, \omega + \psi_-(\xi), \xi) - \sin(\omega + \psi_-(\xi)) - \rho\frac{\partial\psi}{\partial\xi}(\xi).h(\rho, \omega + \psi_-(\xi), \xi), \\ \xi' = \rho h(\rho, \omega + \psi_-(\xi), \xi). \end{cases}$$

Then

$$g(\rho, \omega + \psi_-(\xi), \xi) = a(\xi) + \rho b(\omega + \psi_-(\xi), \xi) + O(\rho^2),$$

so $g(0, \psi_-(\xi), \xi) = \sin(\psi_-(\xi)) = a(\xi)$. So (3.3) is equivalent to

$$\begin{cases} \rho' = \lambda(\xi)\rho(1 + O(|\rho| + |\omega|)), \\ \omega' = \beta(\xi)\rho - \lambda(\xi)\omega + O((|\rho| + |\omega|)^2), \\ \xi' = \rho(\gamma(\xi) + O(|\rho| + |\omega|)), \end{cases} \tag{3.9}$$

with $\lambda(\xi) = \cos(\psi_-(\xi))$ and β, γ smooth functions. The Jacobian matrix of the right-hand side in (3.9) is

$$\begin{pmatrix} \lambda(\xi) & 0 & 0 \\ \beta(\xi) & -\lambda(\xi) & 0 \\ \gamma(\xi) & 0 & 0 \end{pmatrix}.$$

Let us change coordinates further in order to diagonalize this Jacobian. Consider $\tilde{\omega} = \omega + g_1(\xi)\rho$ and $\tilde{\xi} = \xi + g_2(\xi)\rho$, with g_1 and g_2 to be chosen. One has

$$\tilde{\omega}' = \omega' + \frac{\partial g_1}{\partial \xi}(\xi)\xi'\rho + g_1(\xi)\rho',$$

so $\tilde{\omega}' = (\beta(\xi) + 2g_1(\xi)\lambda(\xi))\rho - \lambda(\xi)\tilde{\omega} + O((|\rho| + |\omega|)^2)$, and by picking $g_1 = -\beta/(2\lambda)$, we obtain what we look for. Indeed, with this change of variables, $O((|\rho| + |\omega|)^k) = O((|\rho| + |\tilde{\omega}|)^k)$ for all k; moreover,

$$\tilde{\xi}' = \xi' + \frac{\partial g_2}{\partial \xi}(\xi)\xi'\rho + g_2(\xi)\rho' = \rho(\gamma(\xi) + g_2(\xi)\lambda(\xi)) + O((|\rho| + |\omega|)^2),$$

and we choose $g_2 = -\gamma/\lambda$. In these new variables,

$$\begin{cases} \rho' = \lambda(\tilde{\xi})\rho(1 + O(|\rho| + |\tilde{\omega}|)), \\ \tilde{\omega}' = -\lambda(\tilde{\xi})\tilde{\omega} + O((|\rho| + |\tilde{\omega}|)^2), \\ \tilde{\xi}' = \rho\, O(|\rho| + |\tilde{\omega}|). \end{cases}$$

A smooth change of time finishes the proof. ∎

PROPOSITION 3.8 ($\mathscr{C}^\infty$-normal form). *Set $\Omega = \rho\omega$; there exist A, B, C smooth functions on a neighborhood of $\{0\} \times D$ such that the vector field Y in* (3.8) *is smoothly conjugated to*

$$Y^\infty : \begin{cases} \rho' = -\rho(1 + \Omega A(\Omega, \xi)), \\ \omega' = \omega(1 + \Omega B(\Omega, \xi)), \\ \xi' = \Omega C(\Omega, \xi). \end{cases} \tag{3.10}$$

We postpone the proof of Proposition 3.8 to the end of the section and proceed to prove Theorem 3.3.

LEMMA 3.9. *For z_0 in S^s, the contact point $z_\Sigma(z_0)$ and the contact time $t_\Sigma(z_0)$ with Σ are well defined. These two mappings are smooth functions of z_0 on S^s.*

Proof. Thanks to Proposition 3.8, the globally invariant manifold S^s, fibered by stable manifolds, is straightened to $\{\omega = 0,\ \rho \geq 0\}$. But on $\{\omega = 0\}$, we have the trivial dynamics

$$\begin{cases} \rho' = -\rho, \\ \omega' = 0, \\ \xi' = 0. \end{cases}$$

In particular, associating to an initial condition $z_0 = (\rho_0, 0, \xi_0)$ on the stable stratum, the contact point with Σ is simply projecting to $(0, 0, \xi_0)$, so the mapping $z_0 \mapsto z_\Sigma(z_0)$ is smooth on S^s. Moreover, $\rho(v) = e^{-v}\rho_0$, where v is the new time $\mathrm{d}t = \rho\,\mathrm{d}v/(r\lambda(\xi))$ (see the proof of Proposition 3.8). As a result,

$$t_\Sigma(z_0) = \int_0^\infty \frac{\rho(v)}{r(\xi(v))\lambda(\xi(v))}\,\mathrm{d}v = \frac{\rho_0}{r(\xi_0)\lambda(\xi_0)},$$

which is also smooth on S^s. ◻

Let $t \in [0, t_f]$, and let z_0 be close enough to the initial condition $\overline{z}_0$ of the reference extremal. If z_0 does not belong to S^s, the extremal curve $z(t, z_0)$ does not meet Σ and is well defined. If (t, z_0) belongs to $([0, t_f] \times S^s)\backslash\Delta$, either $t < t_\Sigma(z_0)$ and $z(t, z_0)$ is again the value at t of the smooth extremal curve or $t > t_\Sigma(z_0)$, and our analysis shows that $z(t, z_0)$ is uniquely defined and such that

$$z(t, z_0) = z(t - t_\Sigma(z_0), z_\Sigma(z_0)).$$

It is clear that this construction leads to a flow which is smooth on $[0, t_f] \times S^0$ and also on $([0, t_f] \times S^s)\backslash\Delta$ thanks to Lemma 3.9. To conclude the proof of Theorem 3.3, it remains to prove that the flow is continuous on $O_{\overline{z}_0}$.

Let z_0 and z_1 belong to $O_{\overline{z}_0}$, with $z_0 \in S^s$ and two close times t_0, t_1 in $[0, t_f]$, and O_δ be a small neighborhood of $\overline{z} := z_\Sigma(\overline{z}_0)$. The extremal from z_0 is passing through $z_\Sigma(z_0) \in O_\delta$. We want to control the quantity $|z(t_1, z_1) - z(t_0, z_0)|$. Suppose, without loss of generality, that $t_0, t_1 > t_\Sigma(z_0)$. Let $\varepsilon > 0$, and denote t_δ the contact time with O_δ of the extremal starting from z_0, *respectivly*, t'_δ the exit time from this neighborhood. We can choose z_0 and z_1 to be close enough, so that $|z(t_\delta, z_0) - z(t_\delta, z_1)| < \varepsilon/3$, simply because the flow is continuous when the singular locus is not crossed yet. We will use the following lemma, which gives a uniform bound on the time interval spent by extremals in a neighborhood of $\overline{z}$ to conclude (the result appears in [1]; we give an alternative proof).

LEMMA 3.10 ([1]). *For all $\delta > 0$, there exists a neighborhood O_δ of $\overline{z}$ in which every extremal spends a time smaller than δ.*

Proof. We will prove it in a neighborhood of $\overline{z}_-$, the situation being symmetric around $\overline{z}_+$. Let us define $O_{\delta-} = \{z \mid \rho < \delta,\ |\theta - \theta_-| < \delta\}$, on which $z \mapsto \cos\theta \cdot H_{01}(z) + \sin\theta \cdot H_{02}(z)$ is smooth and thus bounded. Now set

$$M_\delta = \sup_{z \in O_{\delta-}} \cos\theta \cdot H_{01}(z) + \sin\theta \cdot H_{02}(z),$$

and remember it is negative on $O_{\delta-}$. Then, for any extremal in $O_{\delta-}$, $\dot{\rho}(s) \leq M_\delta \rho(s)$, which implies

$$\rho(s) \leq \rho(0) e^{M_\delta s}.$$

So we bound the time spent in $O_{\delta-}$ by

$$\Delta_{O_{\delta-}} t \leq \int_0^{\infty} \rho(0) e^{M_\delta s}\,\mathrm{d}s = -\frac{\delta}{M_\delta}\cdot$$

As M_δ tends to a negative value when δ tends to zero, this quantity tends to 0 when δ does, so every extremal spends an arbitrarily small time in that set. The same holds for the time interval $\Delta_{O_{\delta+}} t$ in a similarly defined neighborhood of $\overline{z}_+$, say, $O_{\delta+}$. This settles the case of extremals going through the singular locus. Let z be an extremal without singularity. The time interval in a neighborhood O_δ can be expressed as

$$\Delta_{O_\delta} t = \Delta_{O_{\delta-}} t + \int_{s_-}^{s_+} \rho\,\mathrm{d}s + \Delta_{O_{\delta+}} t$$

as z exits $O_{\delta-}$ at time s_- and enters $O_{\delta+}$ at time s_+ (see Figure 2). To tackle the central part of this sum, let us go back to system (3.10),

$$Y^\infty : \begin{cases} \rho' = -\rho(1 + \Omega A(\Omega, \xi)), \\ \omega' = \omega(1 + B(\Omega, \xi)), \\ \xi' = \Omega C(\Omega, \xi), \end{cases}$$

which, by a time rescaling, is equivalent to

$$Y^\infty : \begin{cases} \rho' = -\rho(1 + \Omega \tilde{A}(\Omega, \xi)), \\ \omega' = \omega, \\ \xi' = \Omega \tilde{C}(\Omega, \xi), \end{cases}$$

with $\tilde{A}$ standing for $(A-B)/(1+\Omega B)$, and $\tilde{C}$ for $C/(1+\Omega B)$. Because of the change of coordinates $\tilde{\omega} = \omega + g_1(\xi)\rho$, when $s = s_-$, one has $\tilde{\omega}_\pm := \tilde{\omega}(s_\pm) = \delta + g_1(\xi_\pm)\rho(s_\pm)$. Let us denote s_2 the new time, with $\mathrm{d}s_2 = \mathrm{d}s/(r\lambda(1+\Omega B))$. The denominator is bounded below by a positive constant in O_δ, so $s_+ - s_- \leq M(s_{2+} - s_{2-})$. We also have $s_{2+} - s_{2-} = \ln(\tilde{\omega}_+/\tilde{\omega}_-)$. So

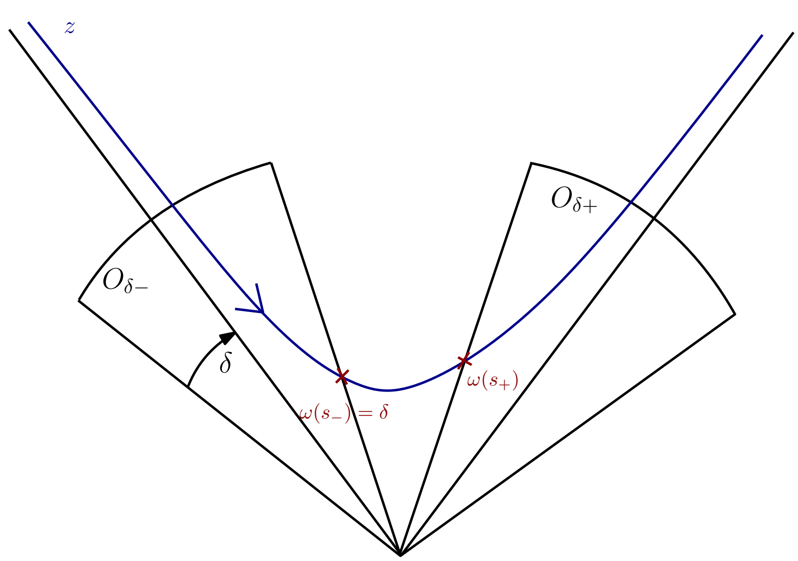

FIG. 2. *Extremal entering O_δ with $\theta(s_-) = \theta_- + \delta$, that is, $\omega(s_-) = \delta$.*

$$\int_{s_-}^{s_+} \rho \, \mathrm{d}s \le M\delta \ln \left(\frac{\delta + g_1(\xi_+)\rho_+}{\delta + g_1(\xi_-)\rho_-} \right),$$

which tends to 0 whenever δ does. This concludes the proof of the lemma. ∎

Then, with a good choice of O_δ, $|z(t'_\delta, z_0) - z(t_\delta, z_0)|$ and $|z(t'_\delta, z_1) - z(t_\delta, z_1)|$ are smaller than $\varepsilon/3$, so that $|z(t'_\delta, z_0) - z(t'_\delta, z_1)| \le |z(t'_\delta, z_0) - z(t_\delta, z_0)| + |z(t_\delta, z_0) - z(t_\delta, z_1)| + |z(t_\delta, z_1) - z(t'_\delta, z_1)| \le \varepsilon$. Now notice that $z(t_0, z_0) = z(t_0 - t'_\delta, z(t'_\delta, z_0))$, and use the regularity of the system when the singular locus is not crossed to conclude the proof of Theorem 3.3.

Remark 3.11. In the case $\overline{z} \in \Sigma_-$, we can compute the jump on the control at the switching time $\overline{t}$ in terms of Poisson brackets:

$$u(\overline{t}_+) - u(\overline{t}_-) = \frac{2\sqrt{r^2(\overline{z}) - \overline{H}_{12}^2}}{r^2(\overline{z})} (\overline{H}_{01}, \overline{H}_{02}). \tag{3.11}$$

Proof of Proposition 3.8. The argument is based on a generalization of the Poincaré–Dulac theorem. Denote H^l the space of homogeneous polynomials of degree l in $\mathbf{R}^n$ with smooth coefficients in $\xi \in \mathbf{R}^k$. We recall that for a linear vector field X that does not depend on ξ (and has no component in the ξ direction), $H^l = \mathrm{im}[X, .]_{|H^l} + \ker[X, .]_{|H^l}$. A vector field Z is said to be resonant with X if $Z \in \ker[X, .]$.

LEMMA 3.12. *Let $X(x, \xi)$ be a smooth vector field in $\mathbf{R}^n \times \mathbf{R}^k$, $X(0, \xi) = 0$. Denote by X_1 its linear part. Then, if X_1 does not depend on ξ, there exist $g_i \in H^i \cap \ker[X_1, .]$, $i = 2, \dots, l$, and a smooth vector field R_l with zero l-jet such that, in a neighborhood of zero, X is smoothly conjugate to*

$$X_1 + g_2 + \cdots + g_l + R_l, \; l \in \mathbf{N}.$$

Proof. We will follow [23] and reason by induction on l, then treat the case $l = \infty$. For $l = 1$, the result is trivial: $X = X_1 + R_1$, where R_1 has zero first jet at zero. Suppose by induction that $g_1, \dots, g_{l-1}$, and R_{l-1} are as desired, $l \ge 2$; R_{l-1} has a zero $l-1$ jet (at zero) and can be written as

$$R_{l-1} = [X_1, Z] + g_l + R_l,$$

where $Z \in H^l$, $g_l \in \ker[X_1, .]$, and R_l is a smooth vector field with zero l-jet. Now

$$[X, Z] = [X_1, Z] + \sum_{i=2}^{l} [g_i, Z] + [R_{l-1}, Z] = [X_1, Z] + R'_l,$$

where R'_l has zero l-jet. Note ϕ_Z the flow of Z, and consider $X^t := (\phi_Z^t)_* X$. One has

$$\frac{\mathrm{d}X^t}{\mathrm{d}t} = [X, Z] = [X_1, Z] + R'_l,$$

so that $X^t = X^0 + t[X_1, Z] + R_{l,t}$ with $j^l(R_{l,t})(0) = 0$. Since Z is a homogeneous polynomial of degree l, it has zero $l-1$ jet, so X and X^t have the same $l-1$ jet, which means that

$$X^0 = X_1 + g_2 + \cdots + g_l + [X_1, Z].$$

For $t = -1$, $X^{-1} = X_1 + g_2 + \cdots + g_l + R_{l,-1}$, and ϕ_Z^{-1} conjugates the two vector fields, which ends the proof by induction. The above construction provides a sequence of formal diffeomorphisms $\varphi_l = \phi_Z^{-1}$ ($Z \in H^l$) such that $(\varphi_l)_* X = X_1 + g_2 + \cdots + g_l + R_l$. Also notice that φ_l and φ_{l+1} have the same l-jet. This defines a sequence of coefficients $g_l(\xi)$ for all l. ∎

By a generalization of Borel theorem proved by Malgrange in [15], we know that there exists a smooth function φ such that the l-jet of φ_l and φ are identical for all $l \in \mathbf{N}$. We can also realize, using the same theorem, the formal series given by the resonant monomials by a smooth vector field X^∞. Thus, we have $\varphi_*(X) = X^\infty + R^\infty$, where R^∞ has zero infinite jet. We begin by looking for monomials that are resonant with the linearized vector field of Y:

$$Y_1 = -\rho\frac{\partial}{\partial\rho} + \omega\frac{\partial}{\partial\omega}$$

(monomials X for which $[Y_1, X] = 0$). The Lie bracket with Y_1 treats ξ as a constant: The map $X \mapsto [Y_1, X]$ is linear in ξ. There are three different cases for such monomials and

$$[Y_1, a(\xi)\rho^i\omega^j\frac{\partial}{\partial\rho}] = (i-j-1)a(\xi)\rho^i\omega^j\frac{\partial}{\partial\rho}\,,$$

$$[Y_1, b(\xi)\rho^i\omega^j\frac{\partial}{\partial\omega}] = (i+1-j)b(\xi)\rho^i\omega^j\frac{\partial}{\partial\omega}\,,$$

$$[Y_1, c(\xi)\rho^i\omega^j\frac{\partial}{\partial\xi}] = (i-j)c(\xi)\rho^i\omega^j\frac{\partial}{\partial\xi}.$$

Setting $\Omega := \rho\omega$, the monomials we are looking for are

$$a(\xi)\rho\Omega^k\frac{\partial}{\partial\rho}\,,\quad b(\xi)\omega\Omega^k\frac{\partial}{\partial\omega}\,,\quad c(\xi)\Omega^k\frac{\partial}{\partial\xi},\quad k \in \mathbf{N}.$$

The lemma allows us to state that the infinite jet of Y can be formally developed on the resonant monomials, so Y is formally conjugate to

$$W : \begin{cases} \rho' = -\rho(1 + \sum_{i\geq 1} a_i(\xi)\Omega^i), \\ \omega' = \omega(1 + \sum_{i\geq 1} b_i(\xi)\Omega^i), \\ \xi' = \rho\sum_{i\geq 1} c_i(\xi)\Omega^i. \end{cases}$$

Notice that, by putting Ω in factor in the formal series above, we get a formal field of the required form (3.10). There exists Y^∞, a smooth vector field on O, such that $W = Y^\infty + R^\infty$, where R^∞ is a smooth function with zero infinite jet along D. At this stage, we have that Y is smoothly equivalent to $Y^\infty + R^\infty$. The last step consists in killing the flat perturbation R^∞. This can be achieved by the path's method: Instead of looking for a diffeomorphism sending $Y_0 := Y$ on $Y_1 := Y^\infty + R^\infty$, we search for a one-parameter family (path) of diffeomorphism $(g_t)_t$ such that

$$g_t^* Y_0 = Y_t, \tag{3.12}$$

Y_t being a path of vector fields joining Y_0 and Y_1. Consider the linear path $Y_t = (1-t)Y_0 + tY_1$, $t \in [0,1]$. Differentiating (3.12) with respect to t, we get

$$\frac{\partial}{\partial t}(g_t^* Y_0) = \dot{Y}_t = Y_1 - Y_0 = R^\infty. \tag{3.13}$$

The family g_t defines a family of vector fields Z_t by

$$Z_t(g_t(x)) = \frac{\partial}{\partial t}g_t(x),$$

and conversely we obtain the desired path of diffeomorphisms by integrating these fields. As a consequence, (3.13) can be rewritten

$$[Y_t, Z_t] = R^\infty. \tag{3.14}$$

We just showed that getting rid of the flat perturbation R^∞ boils down to finding a solution to (3.14). It has been proved in [21] (see Theorem 10) that this equation has a solution. This ends the proof of Proposition 3.8. ☐

4. Regular-singular transition. The existence of a stratification of the flow in the Σ_- case raises the question of the transition: How does the flow behave when one is getting close to the stratum S^s? We answer this question by considering the Poincaré map between two well-chosen sections. Using the normal form given by Proposition 3.8, it is possible to make a precise statement: For given ρ_0 and ω_f, both positive, consider the two sections $\Pi_0 \subset \{\rho = \rho_0\}$ and $\Pi_f \subset \{\omega = \omega_f\}$. As Π_0 is transverse to $\{\omega = 0\}$, it can be parameterized by (ω, ξ) coordinates. Similarly, Π_f is transverse to $\{\rho = 0\}$ and can be parameterized by (ρ, ξ) coordinates (see Figure 3).

THEOREM 4.1. *Let $T : \Pi_0 \to \Pi_f$ be the Poincaré mapping between the two sections, $T(\omega_0, \xi_0) = (\rho(\omega_0, \xi_0), \xi(\omega_0, \xi_0))$. There exist smooth functions P and X defined on a neighborhood of $\{(0,0)\} \times D$ such that*

$$T(\omega_0, \xi_0) = (P(\omega_0 \ln \omega_0, \omega_0, \xi_0), X(\omega_0 \ln \omega_0, \omega_0, \xi_0)).$$

Remark 4.2. The mapping T belongs to the log-exp category [24]. (See [8] for the role of this category in sub-Riemannian geometry.)

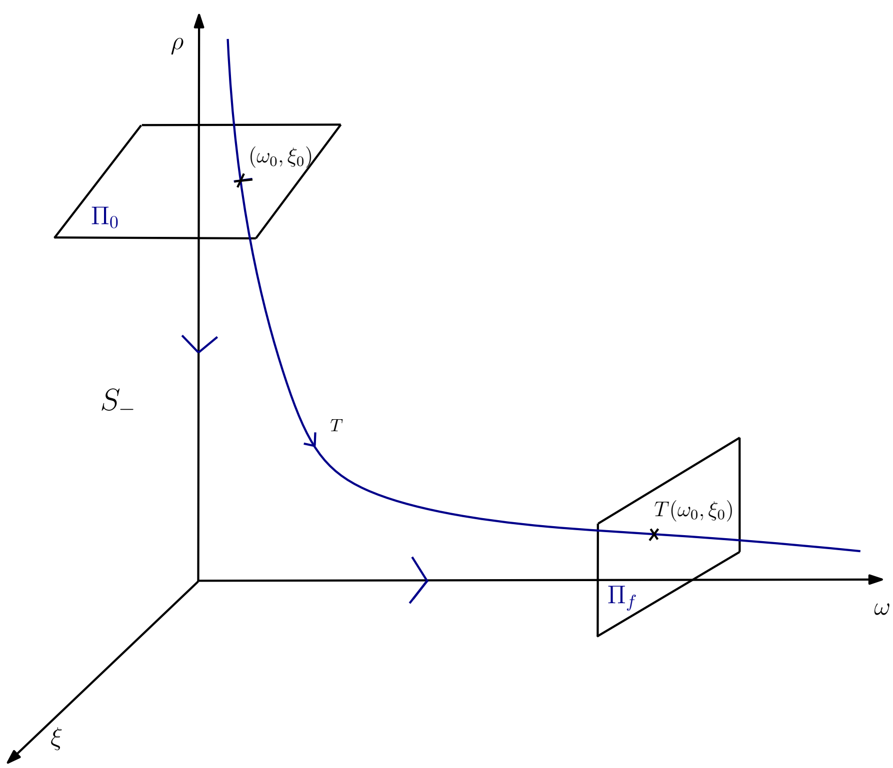

FIG. 3. *Transition map between the two sections.*

Proof. The system (3.10) is equivalent to

$$\begin{cases} \omega' = \omega, \\ \rho' = -\rho(1 + \Omega\tilde{A}(\Omega,\xi)), \\ \xi' = \Omega\tilde{C}(\Omega,\xi). \end{cases} \tag{4.1}$$

It has the same trajectories and thus the same Poincaré mapping between the two sections. The transition time is given by the first equation: $s(\omega_0) = \ln(\omega_f/\omega_0)$. (The singular-regular transition occurs when $\omega_0 \to 0$, and the transition time tends to infinity.) Still noting $u = \rho\omega$, (4.1) implies

$$\begin{cases} \Omega' = -\Omega^2\tilde{A}(\Omega,\xi), \\ \xi' = \Omega\tilde{C}(\Omega,\xi), \end{cases} \tag{4.2}$$

which we want to integrate from an initial condition on Π_0 in time $s(\omega_0)$. We extend this system by the trivial equation $\omega_0' = 0$ and denote φ its associated flow. Then $T(\omega_0,\xi_0) = \varphi(\ln(\omega_f/\omega_0),\omega_0,\rho_0\omega_0,\xi_0)$ (remember that on Π_0, $\Omega_0 = \rho_0\omega_0$). It is not the form we are looking for since $\ln(\omega/\omega_0)$ is not regular at $\omega_0 = 0$, but we have the following estimate on the u coordinate of the flow.

LEMMA 4.3. *There exists a constant M such that, for small enough $\omega_0 > 0$, $\xi \in D$, and integration time $t \leq \ln(\omega_f/\omega_0)$,*

$$0 \leq \Omega(t,\omega_0,\rho_0\omega_0,\xi_0) \leq M\omega_0.$$

Proof. Compare the dynamics of Ω in (4.2) with $v' = -v^2$, which integrates according to

$$v(t,v_0) = \frac{v_0}{1+v_0 t}$$

for $v_0 > 0$. So

$$v(\ln(\omega_f/\omega_0),\rho_0\omega_0) = \frac{\rho_0\omega_0}{1+\rho_0\omega_0\ln(\omega_f/\omega_0)};$$

hence, the estimate as $\omega_0\ln(\omega_f/\omega_0)$ is small enough for small enough $\omega_0 > 0$. □

Let us make a change of time and consider the following rescaled system:

$$\begin{cases} \omega_0' = 0, \\ \Omega' = -(\Omega^2/\omega_0)\tilde{A}(\Omega,\xi), \\ \xi' = (\Omega/\omega_0)\tilde{C}(\Omega,\xi). \end{cases} \tag{4.3}$$

For $\omega_0 > 0$, its flow $\tilde{\varphi}$ is well defined, and the Poincaré mapping is obtained by evaluating it in time $\omega_0\ln(\omega_f/\omega_0)$:

$$T(\omega_0,\xi_0) = \tilde{\varphi}(\omega_0\ln(\omega_f/\omega_0),\omega_0,\rho_0\omega_0,\xi_0).$$

We make a blowup on $\{\Omega = \omega = 0\}$ to prove that T has the required regularity. Set $f(\Omega,\omega,\xi) = (\eta,\omega,\xi)$ with $\eta = \Omega/\omega$: in coordinates (ω,η,ξ), the pulled-back system writes

$$Z : \begin{cases} \omega_0' = 0, \\ \eta' = -\eta^2\tilde{A}(\eta\omega_0,\xi), \\ \xi' = \eta\,\tilde{C}(\eta\omega_0,\xi). \end{cases} \tag{4.4}$$

The vector field Z is actually smooth. The blowup map f sends the cone $-\eta_0\omega \leq \Omega \leq \eta_0\omega$ onto the rectangle $-\eta_0 \leq \eta \leq \eta_0$, $-\omega_0 \leq \omega \leq \omega_0$. According to the previous lemma, we only need to evaluate its flow $\hat{\varphi}(t,\omega_0,\eta_0,\xi_0)$ on a band $\omega_0 \in [-\omega_1,\omega_1]$, $\eta_0 \in [-M,M]$, $\xi \in D$, to compute $\tilde{\varphi}$ in time $\omega_0 \ln(\omega_f/\omega_0)$. As $\hat{\varphi} = (\hat{\eta},\hat{\xi})$ is smooth on such a band, we eventually get

$$T(\omega_0,\xi_0) = (\hat{\eta}(\omega_0 \ln(\omega_f/\omega_0),\omega_0,\rho_0,\xi_0),\hat{\xi}(\omega_0 \ln(\omega_f/\omega_0),\omega_0,\rho_0,\xi_0)),$$

which has the desired regularity. ∎

5. Application to mechanical systems. In this section, we particularize our study to mechanical systems associated with a potential. We go further in the specific case of the potential of the restricted three-body problem. Let Q be an open subset of the plane $\mathbf{R}^2$, let $g : TQ = Q \times \mathbf{R}^2 \to \mathbf{R}^2$ be a smooth function on the (trivial) tangent bundle, and consider the following controlled mechanical system on $M = TQ$:

$$\ddot{q}(t) + g(q(t),\dot{q}(t)) = u(t), \quad u_1^2(t) + u_2^2(t) \leq 1. \tag{5.1}$$

A simple rescaling allows to take into account a more general constraint on the control, $|u(t)| \leq \varepsilon$, for any positive ε. Given a smooth potential $V : Q \to \mathbf{R}$, an important particular case of such a mechanical system is obtained for $g(q,v) = \nabla V(q)$, in which case g is independent of the velocity coordinate v (this corresponds to a system with kinetic energy $\frac{v^2}{2}$). System (5.1) is control-affine of the form (2.1) studied in the previous sections with

$$F_0(q,v) = v\frac{\partial}{\partial q} - g(q,v)\frac{\partial}{\partial v}, \quad F_1(q,v) = \frac{\partial}{\partial v_1}, \quad F_2(q,v) = \frac{\partial}{\partial v_2}\cdot$$

PROPOSITION 5.1. *Minimum time controls of* (5.1) *are piecewise smooth with a finite number of singularities. At such singularities, the control rotates instantaneously of an angle π (generating so-called π-singularities* [10]*).*

Proof. One checks that $\{F_1,F_2,F_{01},F_{02}\}$ have rank four at any point of M, so assumption (A) holds. As $H_{12} = 0$ everywhere, $\Sigma = \Sigma_-$: contacts of minimum time extremals with Σ are isolated according to Proposition 3.1, so there are finitely many of them, and at such points, $u(\bar{t}+) = -u(\bar{t}-)$ by virtue of (3.11). ∎

Theorems 3.3 and 4.1 also apply. Every initial condition leading to a π-singularity has a neighborhood that can be stratified, with a codimension one stratum leading to neighboring π-singularities. The extremal flow is continuous on this neighborhood, and crossing this stratum generates logarithmic (in the sense of Theorem 4.1) singularities on the flow.

An important application is the minimum time control of the elliptic restricted three-body problem [22]. Let $\mu \in (0,1)$ be the ratio of the two primary masses, and let $e \in [0,1)$ be the eccentricity; the elliptic restricted problem seeks the trajectory of a third body whose motion is influenced by the two primary masses, while these masses are not influenced by the third negligible one. The two primaries are assumed to be on elliptic orbits of common eccentricity $e \in [0,1)$ and common focus equal to their center of mass set at the origin (see Figure 4). Let us denote q^1 and q^2 the positions, depending on time, of the two primaries. Let $\mathscr{Q}$ denote the open subset of the time-position space

$$\mathscr{Q} = \{(t,q) \in \mathbf{R} \times \mathbf{R}^2 \mid q \neq q^1(t) \text{ and } q \neq q^2(t)\}$$

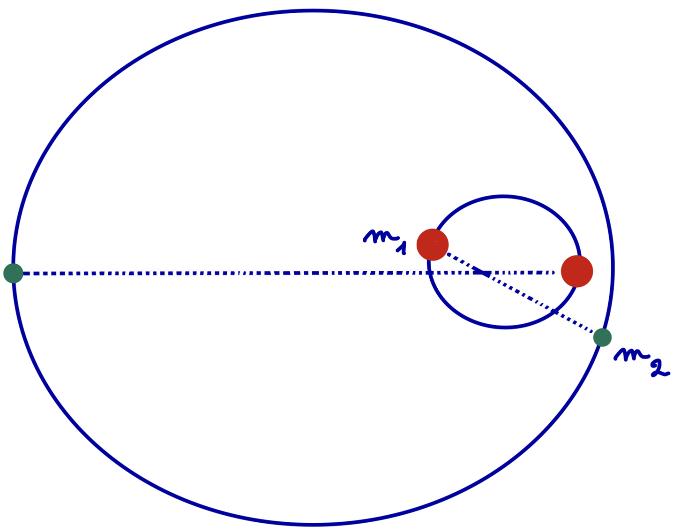

FIG. 4. *Elliptic restricted three-body problem. The two primary masses, m_1 and m_2, are on elliptic orbits of common eccentricity and common focus (located at their center of mass).*

outside collisions. On $\mathscr{Q} \times \mathbf{R}^2$, the dynamics describing the controlled motion of the third body are

$$\ddot{q}(t) + \nabla_q V_{\mu,e}(t, q(t)) = u(t), \tag{5.2}$$

where the time-dependent potential is

$$V_{\mu,e}(t,q) = \frac{1-\mu}{|q - q^1(t)|} + \frac{\mu}{|q - q^2(t)|}.$$

Note that q^1 and q^2 depend on the eccentricity e prescribing the motion of the two primaries, hence the dependence of the potential both on μ and e. A remarkable situation occurs when $e = 0$. The two primaries are in circular motion around their center of mass, and the system can be made autonomous by going into the moving frame attached to the rotating bodies. In this frame, still denoting q the position (now related to moving axes), the dynamics are exactly of the form (5.1) with

$$g(q,v) = (1-\mu)\frac{(q_1+\mu, q_2)}{[(q_1+\mu)^2 + q_2^2]^{3/2}} + \mu\frac{(q_1 - 1 + \mu, q_2)}{[(q_1 - 1 + \mu)^2 + q_2^2]^{3/2}} - q + 2(-v_2, v_1)$$

and $Q = \mathbf{R}^2 \backslash \{(-\mu, 0), (1-\mu, 0)\}$. We refer the reader to [9] for further details on the controlled circular restricted three-body problem. As stated at the beginning of the section, Proposition 3.1 as well as Theorems 3.3 and 4.1 apply to the minimum time control of the circular restricted problem. In addition, it turns out that Proposition 5.1 remains true for the more general elliptic restricted problem. The proof is based on a direct estimation à la Sturm of the time between two singularities and provides a global bound on the number of these singularities. Let $q : [0, t_f] \to \mathbf{R}^2$ be a minimum time trajectory of the elliptic restricted three-body problem (5.2). Let us denote

$$\delta_1 = \min_{[0,t_f]} |q(t) - q^1(t)|, \quad \delta_2 = \min_{[0,t_f]} |q(t) - q^2(t)|,$$

and

$$\delta_{12} = \frac{\delta_1\delta_2}{[(1-\mu)\delta_2^3 + \mu\delta_1^3]^{1/3}}.$$

PROPOSITION 5.2. *Singularities of minimum time trajectories of the elliptic restricted three-body problem are π-singularities. The number of π-singularities for a minimum time trajectory is bounded above by $\left\lfloor t_f/(\pi\delta_{12}^{3/2})\right\rfloor$, where t_f is the minimum time.*

In the circular restricted case ($e = 0$), this proposition provides a global bound on the number of heteroclinic connections of the system after blowup and time change defined in section 3; each π-singularity is associated with a pair of hyperbolic equilibria $(\bar{z}_-, \bar{z}_+)$, and going from one π-singularity to another generates a connection between $\bar{z}_+^i$ and $\bar{z}_-^{i+1}$. Not surprisingly, this bound is expressed in terms of the distance to the primaries, that is, to the singularities of the original dynamics. An interesting open question is to provide an estimate of the distance to the collisions—and thus a more explicit bound on the number of π-singularities—in terms of the boundary conditions in the position-velocity phase space of the restricted problem. Note also than when $\mu = 0$, we go back to the Kepler problem, and $\delta_1 = \delta_{12}$ is the distance to the collision. The proof the proposition uses the following lemma.

LEMMA 5.3. *Let us consider the minimumtime control of*

$$\ddot{q}(t) + \nabla_q V(t, q(t)) = u(t), \tag{5.3}$$

where V is a smooth potential defined on an open subset $O \subset \mathbf{R}^3$. Let $A(t, q)$ be a symmetric matrix of order n whose entries depend continuously on $(t, q) \in O$ such that

$$A(t, q) \geq \nabla_{qq}^2 V(t, q), \quad (t, q) \in O.$$

The following statement holds. Singularities of minimum time trajectories of (5.3) *are π-singularities. If $\bar{t}_1 < \bar{t}_2$ are two such singularities, if $A(t, q(t)) > \nabla_{qq}^2 V(t, q(t))$ for some $t \in [\bar{t}_1, \bar{t}_2]$, there exists a nontrivial solution of $\ddot{y}(t) + A(t, q(t))y(t) = 0$ that vanishes both at $\bar{t}_1$ and $\bar{t}_2' < \bar{t}_2$.*

Proof. Applying the Pontrjagin maximum principle to (5.3), one gets themaximized (time-dependent) Hamiltonian

$$H(t, q, v, p_q, p_v) = p_q.v - p_v.\nabla_q V(t, q) + |p_v|,$$

and $u = p_v/|p_v|$ whenever p_v is not zero. As the equation on p_v is a second-order linear one,

$$\ddot{p}_v(t) + \nabla_{qq}^2 V(t, q(t))p_v(t) = 0,$$

if p_v and $\dot{p}_v$ vanish simultaneously then p_v is identically zero: This is impossible since $p_q = -\dot{p}_v$ would also vanish, leading to $p = (p_q, p_v)$ identically zero, a proscribed case when minimizing time. So $\dot{p}_v \neq 0$ when $p_v = 0$, and the zeros of p_v are isolated. At such a point, the ratio $p_v/|p_v|$ has opposite left and right limits, resulting in a π-singularity. Sturm's comparison theorem [16] allows to conclude. ∎

Proof of Proposition 5.2. Let

$$A(t, q) = \begin{pmatrix} 1 + \frac{1-\mu}{|q-q^1(t)|^3} + \frac{\mu}{|q-q^2(t)|^3} & 0 \\ 0 & \frac{1-\mu}{|q-q^1(t)|^3} + \frac{\mu}{|q-q^2(t)|^3} \end{pmatrix}.$$

A straightforward calculation shows that

$$\det(A(t,q) - \nabla^2_{qq} V_\mu(t,q)) = 3\left[(1-\mu)\frac{(q_2 - q_2^1(t))^2}{|q-q^1(t)|^5} + \mu\frac{(q_2 - q_2^2(t))^2}{|q-q^2(t)|^5}\right] > 0$$

for (t,q) in $\mathscr{Q}$. By the previous lemma, the interval between two singularities is greater than the time interval between two zeros of the scalar equation

$$\ddot{y}(t) + \left(\frac{1-\mu}{|q(t)-q^1(t)|^3} + \frac{\mu}{|q(t)-q^2(t)|^3}\right) y(t) = 0. \tag{5.4}$$

As

$$\frac{1-\mu}{|q(t)-q^1(t)|^3} + \frac{\mu}{|q(t)-q^2(t)|^3} \leq \frac{1-\mu}{\delta_1^3} + \frac{\mu}{\delta_2^3}\,,$$

a solution of (5.4) cannot have consecutive zeros in an interval of length smaller than

$$\frac{\pi}{\sqrt{\frac{1-\mu}{\delta_1^3} + \frac{\mu}{\delta_2^3}}} = \pi\delta_{12}^{3/2}$$

by Sturm comparison again. ∎

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