Optimal control of slow-fast mechanical systems

Jean-Baptiste Caillau, Lamberto Dell'Elce,
Jean-Baptiste Pomet and Jérémy Rouot

Abstract We consider the minimum time control of dynamical systems with slow and fast state variables. With applications to perturbations of integrable systems in mind, we focus on the case of problems with one or more fast angles, together with a small drift on the slow part modelling a so-called secular evolution of the slow variables. According to Pontrjagin maximum principle, minimizing trajectories are projections on the state space of Hamiltonian curves. In the case of a single fast angle, it turns out that, provided the drift on the slow part of the original system is small enough, time minimizing trajectories can be approximated by geodesics of a suitable metric. As an application to space mechanics, the effect of the J_2 term in the Earth potential on the control of a spacecraft is considered. In ongoing work, we also address the more involved question of systems having two fast angles.

J.-B. Caillau
LJAD, Univ. Côte d'Azur & CNRS/Inria, Parc Valrose, F-06108 Nice e-mail: jean-baptiste.caillau@univ-cotedazur.fr

L. Dell'Elce
Inria Sophia Antipolis Méditerranée, Univ. Côte d'Azur, 2004 route des lucioles, F-06902 Valbonne e-mail: lamberto.dell-elce@inria.fr

J.-B. Pomet
Inria Sophia Antipolis Méditerranée, Univ. Côte d'Azur, 2004 route des lucioles, F-06902 Valbonne e-mail: jean-baptiste.pomet@inria.fr

J. Rouot
École Polytechnique Féminine, 2 rue Sastre, F-10430 Rosières-près-Troyes e-mail: jeremy.rouot@epf.fr

Introduction

We consider the following slow-fast control system on an n-dimensional manifold M:

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \varepsilon F_0(I,\varphi,\varepsilon) + \varepsilon \sum_{i=1}^{m} u_i F_i(I,\varphi,\varepsilon), \quad |u| = \sqrt{u_1^2 + \cdots + u_m^2} \leq 1, \tag{1}$$

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \omega(I) + \varepsilon G_0(I,\varphi,\varepsilon) + \varepsilon \sum_{i=1}^{m} u_i G_i(I,\varphi,\varepsilon), \quad \omega(I) > 0, \tag{2}$$

with $I \in M$, $\varphi \in \mathbf{S}^1$, $u \in \mathbf{R}^m$, and fixed extremities I_0, I_f, and free phases φ_0, φ_f. All the data is periodic with respect to the single fast angle φ, and ω is assumed to be positive on M. Extensions are possible to the case of several phases but resonances must then be taken into account.

In the first section, we focus on systems with a single fast angle. The Hamiltonian system provided by applying Pontrjagin maximum principle is averaged after properly identifying the slow variables. The averaged system turns out to be associated with a metric approximation of the original problem. We apply the method to space mechanics, and show how the J_2 term in the Earth potential is responsible for the asymmetry of the metric. In the second section, we give a preliminary analysis of multiphase averaging for minimum time control problems. The case of two fast angles is considered on a simple example. A crucial step is to define a suitable near-identity transformation of the initial state and costate. This work is related with other methods applicable to slow-fast control systems. (See, *e.g.*, the recent papers [1–3,6].)

1 Metric approximation in the case of a single fast phase

1.1 Averaging the extremal flow

According to Pontrjagin maximum principle, time minimizing curves are projections onto the base space $M \times \mathbf{S}^1$ of integral curves (*extremals*) of the maximized Hamiltonian below:

$$H(I,\varphi,p_I,p_\varphi,\varepsilon) := p_\varphi \omega(I) + \varepsilon K(I,\varphi,p_I,p_\varphi,\varepsilon),$$

$$K := H_0 + \sqrt{\sum_{i=1}^{m} H_i^2},$$

$$H_i(I,\varphi,p_I,p_\varphi,\varepsilon) := p_I F_i(I,\varphi,\varepsilon) + p_\varphi G_i(I,\varphi,\varepsilon), \quad i = 0,\dots,m.$$

There are two types of extremals: abnormal ones that live on the level set $\{H = 0\}$, and normal ones that evolve on nonzero levels of the Hamiltonian. One defines the averaged Hamiltonian $\overline{K}$ as

$$\overline{K} := \overline{H}_0 + \overline{K}_0, \quad \overline{H}_0 := \langle p_I, \overline{F}_0 \rangle,$$

$$\overline{K}_0(I,p_I) := \frac{1}{2\pi}\int_0^{2\pi} \sqrt{\sum_{i=1}^{m} H_i^2(I,\varphi,p_I,p_\varphi = 0,\varepsilon = 0)}\,\mathrm{d}\varphi. \tag{3}$$

It is smooth on the open set Ω equal to the complement of $\overline{\Sigma}^c$ where

$$\Sigma := \{(I,p_I,\varphi) \in T^*M \times \mathbf{S}^1 \,|\, (\forall i = 1,m) : \langle p_I, F_i(I,\varphi,\varepsilon = 0)\rangle = 0\},$$

$$\overline{\Sigma} := \varpi(\Sigma) \quad \varpi : T^*M \times \mathbf{S}^1 \to T^*M.$$

One also defines the open submanifold $M_0 := \Pi(\Omega)$ of M. We assume that M_0 is connex. Under the assumption

(A1) $\mathrm{rank}\{\partial^j F_i(I,\varphi,\varepsilon = 0)/\partial\varphi^j,\ i = 1,\dots,m,\ j \geq 0\} = n$, $(I,\varphi) \in M \times \mathbf{S}^1$,

one is able to express some properties of the averaged Hamiltonian in terms of Finsler metric. (We refer the reader, *e.g.*, to [5] for an introduction to Finsler geometry.)

Proposition 1. *The symmetric part* $\overline{K}_0 : (\Omega \subset) T^*M \to \mathbf{R}$ *of the tensor* $\overline{K}$ *is positive definite and* 1*-homogenous. It so defines a symmetric Finsler co-norm.*

We assume moreover that

(A2) $\overline{K}_0(I, \overline{F}_0^*(I)) < 1$, $I \in M$,

where $\overline{F}_0^*$ is the inverse Legendre transform of $\overline{F}_0$. Under this new assumption, one has

Proposition 2. *The tensor* $\overline{K} = \overline{H}_0 + \overline{K}_0$ *is positive definite and defines an asymmetric Finsler co-norm.*

The geodesics are the integral curves of the Hamiltonian $\overline{K}$ restricted to the level set $\{\overline{K} = 1\}$,

$$\frac{\mathrm{d}I}{\mathrm{d}\tau} = \frac{\partial\overline{K}}{\partial p_I}, \quad \frac{\mathrm{d}p_I}{\mathrm{d}\tau} = -\frac{\partial\overline{K}}{\partial I},$$

$$I(0) = I_0, \quad I(\tau_f) = I_f, \quad \overline{K}(I_0, p_I(0)) = 1,$$

and $\tau_f = d(I_0, I_f)$ for minimizing ones. The convergence properties of the original system towards this metric when $\varepsilon \to 0$ are studied in [4].

1.2 Application to space mechanics

We consider the two-body potential case,

$$\ddot{q} = -\mu \frac{q}{|q|^3} + \frac{u}{M}, \quad |u| \le T_{\max}.$$

Thanks to the super-integrability of the $-1/|q|$ potential, the minimum time control system is slow-fast with only angle (the longitude of the evolving body) if one restricts to the case of transfers between elliptic orbits (μ is the gravitational constant). In the non-coplanar situation, we have to analyze a dimension five symmetric Finsler metric. In order to account for the Earth non-oblateness, we add to the dynamics a small drift F_0 on the slow variables. In the standard equinoctial orbit elements, $I = (a, e, \omega, \Omega, i)$, the J_2 term of order $1/|q|^3$ of the Earth potential derives from the additional potential (r_e being the equatorial radius)

$$R_0 = \frac{\mu J_2 {r_e}^2}{|q|^3} \left(\frac{1}{2} - \frac{3}{4} \sin^2 i + \frac{3}{4} \sin^2 i \cos 2(\omega + \varphi) \right)$$

where φ is the true anomaly. As a result, the system now has two small parameters (depending on the initial condition). One is due to the J_2 effect, the other to the control:

$$\varepsilon_0 = \frac{3 J_2 r_e^2}{2 a_0^2}, \quad \varepsilon_1 = \frac{a_0^2 T_{\max}}{\mu M}.$$

Here, a_0 is the initial semi-major axis, $T_{\max}$ the maximum level of thrust, and M the spacecraft mass. We make a reduction to a single small parameter as follows: Defining $\varepsilon := \varepsilon_0 + \varepsilon_1$ and $\lambda := \varepsilon_0/(\varepsilon_0 + \varepsilon_1)$, one has

$$\begin{aligned} \frac{\mathrm{d}I}{\mathrm{d}t} &= \varepsilon_0 F_0(I, \varphi) + \varepsilon_1 \sum_{i=1}^{m} u_i F_i(I, \varphi), \\ &= \varepsilon \left(\lambda F_0(I, \varphi) + (1 - \lambda) \sum_{i=1}^{m} u_i F_i(I, \varphi) \right). \end{aligned}$$

There are two regimes depending on whether the J_2 effect is small against the control ($\varepsilon_0 \ll \varepsilon_1$ and $\lambda \to 0$) or not ($\varepsilon_0 \gg \varepsilon_1$ and $\lambda \to 1$). The critical ratio on λ can be explicitly computed in metric terms.

Proposition 3. *In the average system of the two-body potential including the J_2 effect, $\overline{K} = \lambda \overline{H}_0 + (1 - \lambda) \overline{K}_0$ is a metric tensor if and only if $\lambda < \lambda_c(I)$ with*

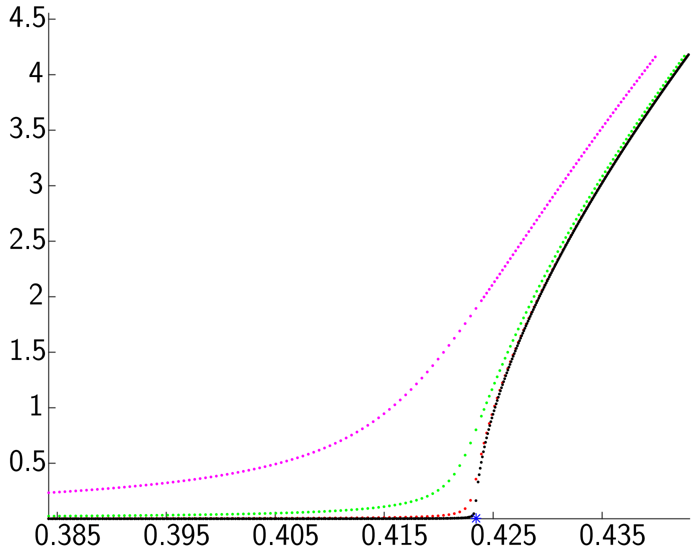

Fig. 1 Value function $\lambda \mapsto \tau_f(\lambda)$, $\tau_d \to 0$ (averaged system). On this example, $a = 30$ Mm, $e = 0.5$, $\omega = \Omega = 0$, $i = 51$ degrees (strong inclination), and $\lambda_c \simeq 0.4239$. The value function is portrayed for $\tau_d = 1e-2, 1e-3, 1e-4, 1e-5$.

$$\lambda_c(I) = \frac{1}{1+\overline{K}_0(I, \overline{F}_0^*(I))}.$$

The relevance of this critical ratio for the qualitative analysis of the original system is illustrated by the numerical simulations displayed in Figures 2 to 4. For a given initial condition I_0 on the slow variables, we let the drift F_0 alone act: We integrate the flow of F_0 during a short positive duration τ_d, then compute the trajectory of the averaged system to go from this point $I(\tau_d)$ back to I_0. For $\lambda < \lambda_c(I_0)$, the tensor $\overline{K}$ is a metric one, and this trajectory is a geodesic. As τ_d tends to zero, the time τ_f to come back from $I(\tau_d)$ tends to zero when $\lambda < \lambda_c(I_0)$. For $\lambda \geq \lambda_c(I_0)$, finiteness of this time indicates that global properties of the system still allows to control it although the metric character of the approximation does not hold anymore. (See Figure 2.) The behaviour of τ_f measures the loss in performance as λ approaches the critical ratio. This critical value depends on the initial condition and gives an asymptotic estimate of whether the thrust dominates the J_2 effect or not. Beyond the critical value, the system is still controllable, but there is a drastic change in performance. As the original system is approximated by the average one, this behaviour is very precisely reproduced on the value function of the original system for small enough ε. (See Figures 3 to 4.)

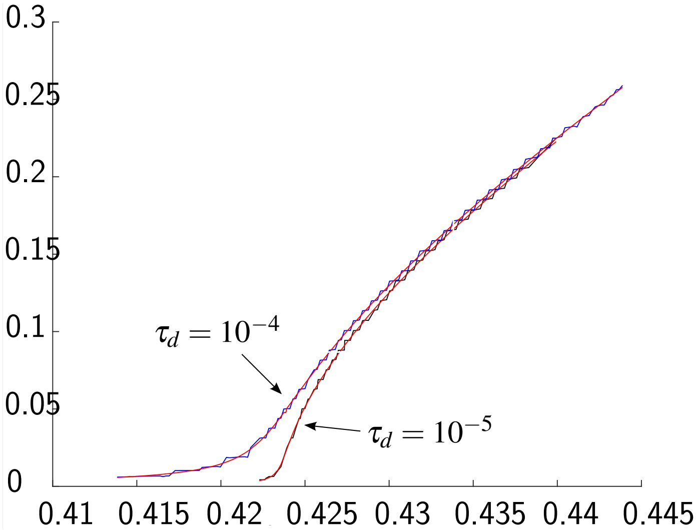

Fig. 2 Value function $\lambda \mapsto \tau_f(\lambda)$, $\tau_d \to 0$ (original system, $\varepsilon = 1e-3$). On this example, $a = 30$ Mm, $e = 0.5$, $\omega = \Omega = 0$, $i = 51$ degrees (strong inclination), and $\lambda_c \simeq 0.4239$. The behaviour of the value function for the original system matches very precisely the behaviour of the averaged one. (See also Figure 4 for a even lower value of ε.)

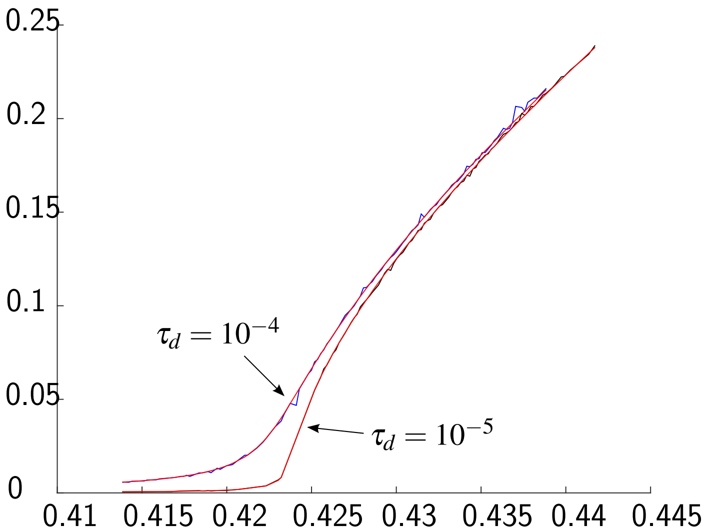

Fig. 3 Value function $\lambda \mapsto \tau_f(\lambda)$, $\tau_d \to 0$ (original system, $\varepsilon = 1e-4$). On this example, $a = 30$ Mm, $e = 0.5$, $\omega = \Omega = 0$, $i = 51$ degrees (strong inclination), and $\lambda_c \simeq 0.4239$. The behaviour of the value function for the original system matches very precisely the behaviour of the averaged one.

2 Averaging control systems with two fast angles

2.1 A simple example

In order to illustrate the our preliminary analysis of multiphase averaging for control systems, we consider an elementary dynamical system consisting of a scalar slow variable, I, and two fast variables, ζ and ψ. The optimal control problem is

$$\begin{aligned}
&\min_{\sqrt{u_1^2+u_2^2}\le 1} t_f \quad \text{subject to}: \\
&\qquad \frac{\mathrm{d}I}{\mathrm{d}t} = \varepsilon\left[\cos\zeta + \cos\left(\zeta-\psi\right)u_1 + u_2\right], \quad \frac{\mathrm{d}\zeta}{\mathrm{d}t} = I, \quad \frac{\mathrm{d}\psi}{\mathrm{d}t} = 1, \\
&\qquad I(0) = I_0, \quad I(t_f) = I_f.
\end{aligned} \tag{4}$$

We note that the frequency of ψ is constant. If one of the two frequencies is non-vanishing on the ambient manifold M, any problem with two frequencies can be recast into this form by means of a change of the time variable, as emphasized in [8]. The Hamiltonian associated to Problem (4) is

$$H = Ip_\zeta + p_\psi + \varepsilon\left[p_I\cos\zeta + |p_I|\sqrt{1+\cos^2\left(\zeta-\psi\right)}\right]. \tag{5}$$

The maximizing control is

$$u_1^{opt} = \frac{p_I}{|p_I|}\frac{\cos\left(\zeta-\psi\right)}{\sqrt{1+\cos^2\left(\zeta-\psi\right)}}, \quad u_2^{opt} = \frac{p_I}{|p_I|}\frac{1}{\sqrt{1+\cos^2\left(\zeta-\psi\right)}},$$

revealing that the sign of p_I determines the direction of the control vector, which imposes a secular drift to the slow variable. Numerical values used in all simulations are $\varepsilon = 10^{-3}$ and $I_0 = \sqrt{2}/2$. Applying averaging theory to the extremal flow of this problem is questionable because the structure of this vector field differs from the one of conventional fast-oscillating systems. As in the case of one fast angle, the equation of motion of p_I includes the term $p_\varphi \partial\omega/\partial I$ that may be of order larger than ε. Hence, adjoints of slow variables are not necessary slow themselves. We justify the application of averaging theory to System (14) by showing that, as in the case of a single fast phase discussed in the previous section, adjoints of fast variables remain ε-small for any extremal trajectory with free phases.

Consider the canonical change of variables $\{I, p_I, \varphi, p_\varphi\} \to \{J, p_J, \psi, p_\psi\}$ such that

$$J = I, \qquad \psi = \Omega(I)\,\varphi, \tag{6}$$

where the matrix-valued function, $\Omega : M \to \mathbf{R}^{2\times 2}$ is defined as

$$\Omega := \frac{1}{\|\omega(I)\|} \begin{bmatrix} \omega_1(I) & \omega_2(I) \\ -\omega_2(I) & \omega_1(I) \end{bmatrix}. \tag{7}$$

Symplectic constraints yield the transformation of the adjoints

$$p_I = p_J + p_\psi \frac{\partial \Omega}{\partial J} \Omega^T \psi, \qquad p_\varphi = p_\psi\, \Omega(J), \tag{8}$$

so that the transformed Hamiltonian is

$$\widetilde{H} = ||\omega(J)|| p_{\psi_1} + \varepsilon \underbrace{K\left(J,\, p_J + p_\psi \frac{\partial \Omega}{\partial J} \Omega^T \psi,\, \Omega^T \psi,\, p_\psi\, \Omega\right)}_{:= \widetilde{K}(J, p_J, \psi, p_\psi)}. \tag{9}$$

Boundary conditions on the adjoints of fast variables require that $p_\varphi(0) = 0$. Evaluating the Hamiltonian at the initial time and normalizing the initial adjoints according to $\|p_{I0}\| = 1$, one sets

$$\varepsilon h := \widetilde{H}(t=0) = \varepsilon \underbrace{K\left(I_0,\, p_{I_0},\, \Omega^T(I_0)\ \psi_0,\, 0\right)}_{O(1)}. \tag{10}$$

Hence, p_{ψ_1} can be evaluated at any time by solving the implicit function

$$p_{\psi_1} = \varepsilon \frac{h - \widetilde{K}(J, p_J, \psi, p_\psi)}{||\omega(J)||} \approx \frac{h - \widetilde{K}(J, p_J, \psi, 0)}{||\omega(J)||} \tag{11}$$

Equation (11) indicates that $p_{\psi_1} = O(\varepsilon)$ when evaluated on a candidate optimal trajectory. As a consequence, p_J has an ε-slow dynamics, *i.e.*

$$\frac{\mathrm{d} p_J}{\mathrm{d} t} = -\underbrace{\frac{\partial \|\omega\|}{\partial J} p_{\psi_1}}_{O(\varepsilon)} - \varepsilon \frac{\partial \widetilde{K}}{\partial J} = O(\varepsilon), \tag{12}$$

which justifies the averaging of the extremal flow. As before, we denote by $\overline{K}$ the averaged Hamiltonian

$$\overline{K} := \frac{1}{4\pi^2} \int_{\mathbf{T}^2} K(I, p_I, \varphi, 0)\ \mathrm{d}\varphi. \tag{13}$$

Here, $p_\varphi = 0$ because the averaging is carried out by considering the limit of the function as ε approaches zero. Averaging the extremal flow yields

$$
\begin{aligned}
\frac{\mathrm{d}\bar{I}}{\mathrm{d}t} &= \varepsilon\frac{\partial\bar{K}}{\partial\overline{p_I}}, & \frac{\mathrm{d}\overline{p_I}}{\mathrm{d}t} &= -\varepsilon\frac{\partial\bar{K}}{\partial\bar{I}} - \overline{p_\varphi}\,\frac{\partial\omega}{\partial\bar{I}}, \\
\frac{\mathrm{d}\varphi}{\mathrm{d}t} &= \varepsilon\frac{\partial\bar{K}}{\partial\overline{p_\varphi}} + \omega\left(\bar{I}\right), & \frac{\mathrm{d}\overline{p_\varphi}}{\mathrm{d}t} &= 0.
\end{aligned}
\tag{14}
$$

2.2 *Near-identity transformation of the initial state and costate*

Changing the initial conditions of averaged trajectories allows one to reduce the drift between $I(t)$ and $\bar{I}(t)$. Qualitatively, one defines a transformation that shifts the initial point of the averaged trajectory to the middle of the short-period oscillations of $I(t)$. The improvement obtained with this expedient is possibly negligible when compared to the estimate provided by Neishtadt theorem for systems with two fast angles [8], which considers the same initial conditions for the two trajectories. Nonetheless, the transformation of the initial variables plays a key role for the optimal control problem. (See [7] for a detailed discussion.) Figure 4 shows that p_I and $\overline{p_I}$ exhibit a steady derive that largely exceeds the expected small drift when the original and averaged systems are integrated with the same initial conditions. In addition, trajectories of the original system strongly depend on the initial angles. We show in the sequel that transforming the adjoints of fast variables is sufficient to drastically reduce the drift of p_I.

The trigger at the origin of the drift of p_I is the wrong assessment of the averaged value of p_φ, as shown in the bottom of Figure 4. This error is of order ε but it induces a steady drift of $\overline{p_I}$ of the same order of magnitude,

$$
\frac{\mathrm{d}\overline{p_I}}{\mathrm{d}t} = \underbrace{-\overline{p_\varphi}\frac{\partial\omega}{\partial\bar{I}}}_{\varepsilon-\text{small error}} - \varepsilon\frac{\partial\bar{K}}{\partial\bar{I}}. \tag{15}
$$

In turn, an ε-small error on $\overline{p_\varphi}$ induces a steady drift of $\overline{p_I}$ that is comparable with its slow motion. Transforming the initial adjoints of fast variables is sufficient to greatly mitigate this problem. More precisely, initial conditions of the averaged and of the original initial value problem are mostly the same, *i.e.*

$$
I(0) = \bar{I}(0) = I_0, \quad p_I(0) = \overline{p_I}(0) = p_{I0}, \quad \varphi(0) = \varphi_0, \tag{16}
$$

except for the adjoints of fast variables, which are such that

$$
\overline{p_\varphi}(0) = \overline{p_{\varphi_0}} \qquad \text{and} \qquad p_\varphi(0) = \overline{p_{\varphi_0}} + \nu_{p_\varphi}\left(I_0, p_{I0}, \varphi_0, \overline{p_{\varphi_0}}\right), \tag{17}
$$

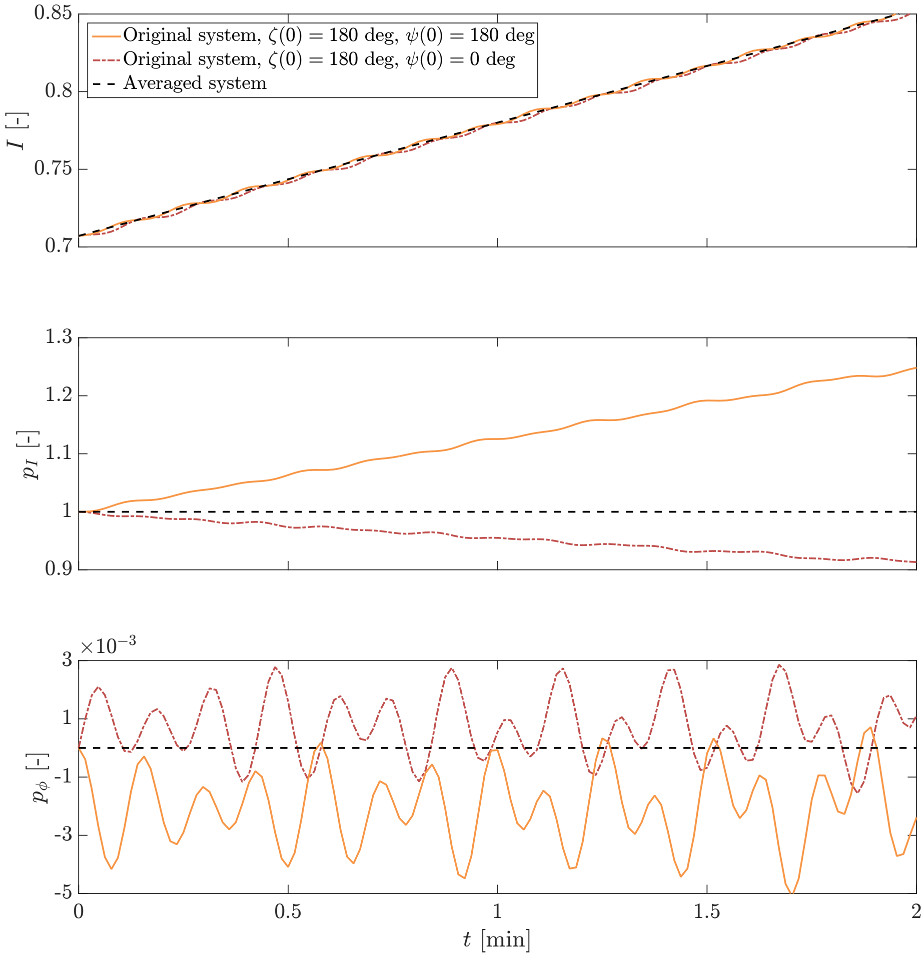

Fig. 4 Numerical integration of the simple example. Trajectories of the original and averaged system emanate from the same point of the phase space. Initial adjoints are $p_I(0) = 1$ and $p_\psi(0) = p_\zeta(0) = 0$.

where, assuming that I_0 is in a non-resonant zone, ν_{p_φ} is given by

$$\nu_{p_\varphi} = -i \sum_{0<|k|\leq N} \frac{e^{ik\cdot\overline{\varphi}}}{k\cdot\omega\left(\overline{I}\right)} \left[-\frac{\partial K}{\partial \varphi}\right]^{(k)}. \tag{18}$$

As a result, p_φ oscillates with zero mean about $\overline{p_\varphi}$, and the drift between $p_I(t)$ and $\overline{p_I}(t)$ is drastically reduced.

Besides, changing p_φ is mandatory to have consistent trajectories of the averaged and original systems. Transforming the initial value of slow variables and their adjoints is less important, but it can further reduce the drift between these trajectories. Reconstructed trajectories (dash-dotted lines) of I

and p_φ well overlap with their original counterpart, see Figure 5. Nevertheless, the reconstruction of p_I is wrong (in the very-specific case of the simple example, $\nu_{p_I} = 0$). Again, the term $p_\varphi \partial\omega/\partial I$ in the dynamics of p_I is responsible for this error. In fact, if short-period variations of p_φ are neglected, the Fourier expansion of the right-hand side is carried out by introducing ε-small errors in the evaluation of the ε-slow dynamics. The transformation of p_I should be carried out by including ν_{p_φ} in the Fourier expansion, namely

$$\nu_{p_I} = -i \sum_{0<|k|\leq N} \frac{e^{ik\cdot\overline{\varphi}}}{k\cdot\omega\left(\overline{I}\right)} \left[-\left(\overline{p_\varphi} + \nu_{p_\varphi}\right)\frac{\partial\omega}{\partial I} - \frac{\partial K}{\partial I} \right]^{(k)}. \tag{19}$$

Ongoing work is concerned with the extension of this analysis to resonant zones. When resonances of rather low order are crossed, one has to patch together resonant and non-resonant normal forms. Detecting properly where to patch these approximations will be the subject of further studies.

Acknowledgements Work supported by the French Space Agency (CNES R&T contract no. R-S13/BS-005-012); by Thales Alenia Space, Inria and UCA-JEDI Investissements d'avenir (ANR-15-IDEX-01, PEPS MSI); by the FMJH Program PGMO and EDF-Thales-Orange (PGMO grant no. 2016-1753H). The authors thank Richard Epenoy (CNES) and Thierry Dargent (Thales Alenia Space) for many insightful exchanges.

References

1. Bombrun, A.; Pomet, J.-B. The averaged control system of fast oscillating control systems. *SIAM J. Control Optim.* **51** (2013), no. 3, 2280–2305.
2. Bonnard, B.; Henninger, H.; Nemcova, J.; Pomet, J.-B. Time versus energy in the averaged optimal coplanar Kepler transfer towards circular orbits. *Acta Appl. Math.* **135** (2015), no. 2, 47–80.
3. Caillau, J.-B.; Dargent, T.; Nicolau, F. Approximation by filtering in optimal control and applications. *IFAC PapersOnLine* **50** (2017), no. 1, 1649–1654. Proceedings of the 20th IFAC world congress, Toulouse, July 2017.
4. Caillau, J.-B.; Pomet, J.-B.; Rouot, J. Metric approximation of minimum time control systems. *HAL preprint* (2017), no. 01672001, 1–12.
5. Dahl, M. F. A brief introduction to Finsler geometry. Lecture notes, 2006. `math.aalto.fi/~fdahl/finsler/finsler.pdf`
6. Dargent, T.; Nicolau, F.; Pomet, J.-B. Periodic averaging with a second order integral error. *IFAC PapersOnLine* **50** (2017), no. 1, 2892–2897. Proceedings of the 20th IFAC world congress, Toulouse, July 2017.
7. Dell'Elce, L.; Caillau, J.-B.; Pomet, J.-B. Averaging optimal control systems with two fast variables *HAL preprint* (2018), no. 01793704, 1–18.
8. Lochak, P.; Meunier, C. *Multiphase averaging for classical systems.* Springer, 1988.

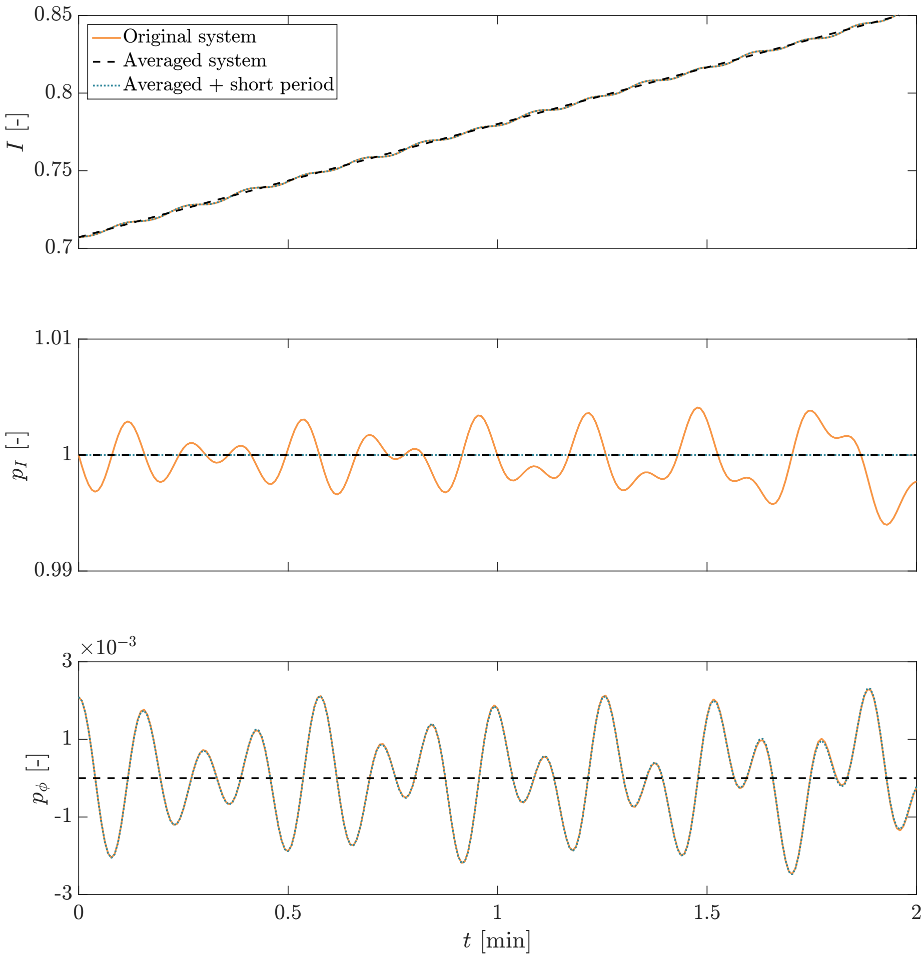

Fig. 5 Reconstruction of short-period variations using an appropriate transformation of the initial state and costate.