

State Constraint Analysis for Powered Descent Guidance

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Abstract: We study the powered-descent guidance problem for soft planetary landing. This optimal control problem seeks the best trajectory for propulsively landing a vehicle at a given point while consuming as little propellant as possible and satisfying constraints on the control (thrust direction) and the state (obstacle avoidance). As such, it is very relevant to applications to reusable launchers. The problem and its variants have been extensively studied using finite-dimensional optimization techniques (*e.g.* convex solvers), and less so with indirect methods. Even the variant with the simplest dynamics is not yet completely understood mathematically, as highlighted by the recent study of Leparoux and Jean (2022). We provide the expression of the control along state-constrained arcs, which allows us to implement the indirect method accounting for the control and state constraints. We then explore the space of trajectory structures by sampling the initial condition. This highlights the richness of the problem, and lets us envision, as future work, low computation MPC-style real-time controllers.

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INTRODUCTION

The powered-descent guidance (PDG) problem for soft planetary landing has received considerable attention in recent years, driven by applications to reusable launchers and robotic missions to celestial bodies. The problem seeks to determine the optimal trajectory that brings a vehicle to a prescribed landing site while minimizing fuel consumption and satisfying constraints on thrust magnitude, thrust direction, and obstacle avoidance. Açıkmeşe and Ploen (2007) pioneered the use of convex optimization techniques for this problem, demonstrating real-time computation capabilities. Lu (2018) develops a complete solution using and indirect method for solving the PDG, but without constraint on the state or the thrust direction. More recently, Spada et al. (2023) proposed a direct-indirect hybrid strategy that combines the computational efficiency of direct methods with the optimality guarantees of indirect approaches, but still without state or control constraint. Lu and Davami (2024) revisited the problem, providing new insights into the structure of optimal solutions.

While finite-dimensional optimization techniques have proven highly successful for real-time implementation, the mathematical structure of the PDG problem with combined control and state constraints remains incompletely understood. The recent work of Leparoux et al. (2022) highlighted open questions regarding the characterization of optimal trajectories, particularly in the presence of a glide-slope state constraint for obstacle and ground avoidance. The present work addresses these gaps by providing an explicit expression for the control along state-constrained arcs and implementing a complete indirect method that accounts for both pointing constraints (on

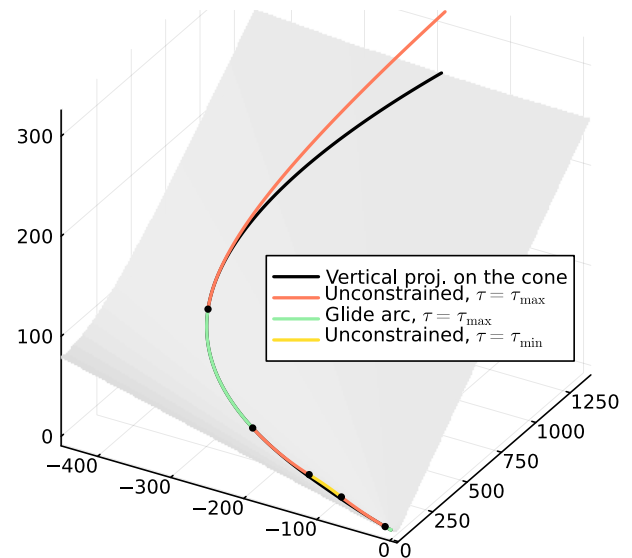


Fig. 1. A sample trajectory with six arcs including two glide arcs (green). The last arc is very short; please zoom in to view it.

thrust direction) and glide-slope constraints (on vehicle position). We systematically explore the space of trajectory structures through extensive numerical experiments, revealing the rich variety of optimal solutions that emerge from different initial conditions. Contrary to what recent work suggests (You et al. (2022)), the state constraint *can* be active along arcs of positive duration. In fact, these long state-constrained arcs appear in 0.75 % of the optimal trajectories computed in our experiment, see Figure 1.

The paper is organized as follows. Section 1 formulates the PDG problem with three-dimensional motion, minimal fuel objective, and combined control and state constraints, then derives the necessary optimality conditions from Pontryagin's maximum principle. Section 2 analyzes the interaction between pointing and glide-slope constraints, providing the key expression for the Lagrange multiplier along state-constrained arcs. Section 3 describes our numerical approach, combining direct optimization for initialization with indirect shooting methods for high-precision solutions. In Section 4 we present extensive numerical experiments revealing 56 distinct trajectory structures. Section 5 analyzes the conditions under which both constraints can be simultaneously active.

1. THE POWERED-DESCENT GUIDANCE PROBLEM

We further the study of the powered-descent guidance (PDG) problem in the following context:

- no drag; we assume the velocity in this late stage is small enough that drag can be ignored
- constant gravity; again, gravity is assumed to not vary significantly along an optimal trajectory
- the control is constrained with a thrust pointing (directional) constraint, and magnitude lower and upper bounds
- glide-slope *state* constraint

The system state is $\mathbf{x} = (r, v, m) \in \mathbb{R}^7$ where r is the position in a planet-fixed frame and v is the velocity. The frame origin is the landing target, on the ground. $m(t)$ is the vehicle mass at time t .

We use subscripts x, y and z for the cartesian coordinates. The z -axis is aligned with gravity and points upwards. For any 3D vector $p = (p_x, p_y, p_z)$, we define $\bar{p} = (p_x, p_y, 0) \in \mathbb{R}^3$, which we sometimes interpret as $(p_x, p_y) \in \mathbb{R}^2$ when convenient. We also define $\hat{p} = \frac{\bar{p}}{\|\bar{p}\|}$.

We consider a *glide-slope* constraint on the state: $h(r) \equiv r_z - \tan \gamma \|\bar{r}\| \geq 0$, where $0 \leq \gamma < \pi/2$. The glide-slope constraint defines an inverted cone with apex at the origin and forces the vehicle to stay above the cone. This models the avoidance of the ground and small hills and boulders around the landing site. We define $n(r) \equiv (-\tan \gamma \hat{r}; 1)$, the (non-unit) normal to the glide-slope cone at any point in the intersection of the cone and the vertical plane through r and the origin (when that plane is unique). The normal vector is not defined when $\bar{r} = 0$ (see remark 2.4 in Leparoux et al. (2022)). Then $h(r) = n(r) \cdot r$ and $n(r) = \nabla h$. When the system moves along a trajectory, we use the shorthand $n(t) \equiv n(r(t))$.

The system's control is $u \in U \subset \mathbb{R}^3$. It is such that Tu is the instantaneous thrust vector. The set U is static, non-convex and defined by two constraints:

- a constraint on the norm of u : $0 < \tau_{\min} \leq \|u\| \leq \tau_{\max} \leq 1$, and
- a constraint on the direction of u : $u_z \geq \|u\| \cos \theta^*$, which we call the *pointing constraint*.

The parameter θ^* can be arbitrary, but some nice properties hold when $\theta^* < \frac{\pi}{2}$ (Section 2), or when $\theta^* \in (\gamma, \pi -$

$\gamma)$ (Section 5). We write the control as $u(t) = \tau(t)d(t)$ where the unit vector d is the thrust direction, and $\tau(t) \in [\tau_{\min}, \tau_{\max}]$ is the throttle. The pointing constraint also writes as $d_z \geq \cos \theta^*$. The powered descent guidance (PDG) problem asks for the control $u(t)$ that minimizes fuel consumption, *i.e.* maximizes the final mass $m(t_f)$. It is defined formally as follows:

$$\begin{aligned} \min_{u(t) \in U} -m(t_f) \quad & \text{where } t_f \text{ is the free final time, subject to} \\ \dot{r}(t) &= v(t), \\ \dot{v}(t) &= \frac{T}{m(t)}u(t) + g = K(t)d(t) + g, \\ \dot{m}(t) &= -\alpha T \|u(t)\| = -\alpha T \tau(t), \\ h(r(t)) &\geq 0, \quad u_z(t) \geq \|u(t)\| \cos \theta^*, \\ \tau_{\min} &\leq \|u(t)\| \leq \tau_{\max}, \\ (r, v, m)(0) &= (r_0, v_0, m_0) \quad \text{and} \quad r(t_f) = v(t_f) = 0. \end{aligned} \quad (1)$$

The maximal thrust that the vehicle can provide is denoted T . The vector g is the local gravity: $g = (0, 0, -g_0)$ with $g_0 > 0$. The constant parameter $\alpha > 0$ adjusts the fuel flow, see Section 4.

Remark 1. The function $K : \mathbb{R}^+ \mapsto \mathbb{R}^+$ above is used to simplify expressions. On an interval with constant $\tau \equiv \|u\|$, it has the following derivatives: $K = T\tau/m$, $\dot{K} = \alpha K^2$, $\ddot{K} = 2\alpha^2 K^3$.

Let $p^0 \in \mathbb{R}$, $\mathbf{p} = (p_r, p_v, p_m) \in \mathbb{R}^7$ be the costate vector. We adjoin the glide slope constraint with multiplier $\mu(t) \geq 0$, to the hamiltonian:

$$\begin{aligned} H(\mathbf{x}, u, \mathbf{p}, \mu) &= H_0 + TH_1, \quad \text{with} \\ H_0 &= p_r \cdot v + p_v \cdot g + \mu h(r) \quad \text{and} \\ H_1 &= \tau \left(\frac{1}{m} p_v \cdot d - p_m \alpha \right). \end{aligned}$$

H does not depend on p^0 since the “running cost” is zero. H_0 is the part of H that does not depend on the control u . From Pontryagin's maximum principle (*e.g.* Bryson, Jr. and Ho (1975); Vinter (2000)), we know that if u is an optimal control, then there exists $p^0 \leq 0$, an adjoint vector $\mathbf{p}$ that is absolutely continuous and $\mu \geq 0$, such that

$$\begin{aligned} \dot{p}_r(t) &= -\frac{\partial H}{\partial r}(t) = -\mu(t) \frac{\partial h}{\partial r}(t) = -\mu(t)n(t), \\ \dot{p}_v(t) &= -\frac{\partial H}{\partial v}(t) = -p_r(t), \\ \dot{p}_m(t) &= -\frac{\partial H}{\partial m}(t) = \frac{T}{m^2(t)} p_v(t) \cdot u(t), \\ \text{supp } \mu(t) &\subset \{t \mid h(r(t)) = 0\}. \end{aligned} \quad (2)$$

On intervals along which the glide slope constraint is not active, p_r is constant and p_v linear. The final time t_f and final mass are free, hence $H(t_f) = -p^0 \frac{\partial(-m)}{\partial t} \Big|_{t=t_f} = 0$ and $p_m(t_f) = p^0 \frac{\partial(-m)}{\partial m} \Big|_{t=t_f} = -p^0 \geq 0$. We assume that optimal trajectories are not abnormal and we can therefore normalize p_0 by choosing $p_0 = -1$.

1.1 Maximum principle

The statements in this paragraph come from Leparoux et al. (2022). The optimal control maximizes H_1 over

$\{(\tau, d) \mid u = \tau d \in U\}$. τ is positive because $\tau \geq \tau_{\min} > 0$. Thus, the optimal control's direction, d , maximizes

$$\Psi(d) = \frac{1}{m} p_v \cdot d - p_m \alpha, \quad (3)$$

the *switch function*, which is linear. Maximizing a linear functional over a bounded set is equivalent to maximizing it over its convex hull. Here, we maximize over the intersection of the unit ball and a half-space. We get

$$d = \begin{cases} \left(\sin \theta^* \widehat{p}_v, \cos \theta^* \right) & \text{if } p_{v,z} \leq \|p_v\| \cos \theta^*, \text{ and we} \\ & \text{say that the pointing constraint} \\ & \text{is active,} \\ \widehat{p}_v & \text{"otherwise."} \end{cases} \quad (4)$$

We assume that the case ($\widehat{p}_v = 0$ and $p_{v,z} \leq 0$) does not happen: it is a measure-zero subset and is never seen in practice. Finally, $\tau = \begin{cases} \tau_{\min} & \text{if } \Psi(d) < 0 \\ \tau_{\max} & \text{if } \Psi(d) > 0 \end{cases}$. Leparoux et al.

show that a singular arc (along which Ψ is zero) may only appear if the trajectory stays in a vertical plane. For our practical purpose, we ignore the possibility of singular arc.

They also show that all optimal trajectories have a Max-(Min or Singular)-Max structure for the throttle τ , wherein each arc can degenerate to nonexistence. The pointing constraint ($d_z \geq \cos \theta^*$) can be active over positive-duration intervals. The glide-slope state constraint ($h(r) \geq 0$) can be active at instantaneous contacts, or along positive-duration intervals.

2. POINTING AND GLIDE-SLOPE CONSTRAINTS

The pointing constraint is neither a state nor a mixed constraint, but it is important to consider it separately because it makes the control set U non-smooth, *i.e.* U has sharp edges generated by the control constraints. Assume that the state constraint is not active, then the position adjoint, p_r , is constant and the velocity adjoint, p_v , follows a straight line motion: $p_v(t) = p_v^0 - t p_r$. The path of $\widehat{p}_v$ is the projection of that line segment onto the unit-sphere S^2 . We see that $\widehat{p}_v$ moves *monotonously* along a half great-circle of S^2 and while the pointing constraint is not active, the thrust direction d equals $\widehat{p}_v$.

When the pointing constraint becomes active, d changes course non-continuously and moves along the horizontal circle $C = \{\|\bar{d}\| = \sin \theta^*, d_z = \cos \theta^*\}$.

An interesting interaction between state and control constraints. Now, let us assume that $\theta^* < \pi/2$. The space of d is then a spherical cap centered at the north pole with half-angle θ^* . The boundary of that cap is the circle C defined above. C is a circle smaller than a great-circle. Therefore $\dot{d}$ always shows a discontinuity when the pointing constraint becomes active (*i.e.* when $\widehat{p}_v(t)$ crosses C southward). More importantly, once $\widehat{p}_v$ exits the inside of the cap, it can not re-enter it, because $\widehat{p}_v$ moves along a half great-circle. The only way for $\widehat{p}_v$ to get back inside (*i.e.*, to deactivate the pointing constraint) is for $p_v(t)$ to change its direction $p_r(t)$, which can happen only by activating the glide slope state constraint, because $\dot{p}_r = -\mu n$ where μ is the multiplier of the state constraint.

The glide-slope state constraint. We now assume that the control pointing constraint is inactive and compute

the control during an arc of positive length along which the glide slope state constraint is active, *i.e.* $h(t) = 0$. In this section, we derive an expression for the multiplier $\mu(t)$ which we summon from the fourth derivative of h . Inside the constrained arc, h and its derivatives are zero.

We have $h(t) \equiv h(r(t)) = 0 = n(t) \cdot r(t)$ where $n(t) \equiv (-\tan \gamma \widehat{r}(t); 1)$. In the expression of n , the vector $\bar{r}$ is normalized, so its time derivative is orthogonal to $\bar{r}$, therefore: $\dot{n} \cdot r = 0$ and, dropping the dependence on t , $\dot{h} = n \cdot v$. So along the constrained arc, we also have $n(t) \cdot v(t) = 0$ (*i.e.* the velocity is in the plane tangent to the cone at $r(t)$). To help in the computation of the next derivatives, we use the auxiliary function f defined in Section A. Then, $\dot{n} = \frac{\tan \gamma}{\|\bar{r}\|^3} f(\bar{r}, \bar{v})$. From (A.3) (Cauchy-Schwartz), we see that $\dot{n} \cdot v \leq 0$.

On the constrained arc: $\ddot{h} = 0 = \dot{n} \cdot v + n \cdot \dot{v} = \dot{n} \cdot v + K n \cdot d - g_0$. We find

$$\ddot{h} = K n \cdot d - (g_0 - \dot{n} \cdot v) \equiv K n \cdot d - G(x) = 0. \quad (5)$$

Note that $G(x) = g_0 - \dot{n} \cdot v$ is positive, so the thrust pushes us, thankfully, away from and above the cone.

Proposition 1. Let $I = (a, b)$, $a < b$, be a time interval along which the glide-slope constraint is active. Then there is no **Switch** junction in I . (A **Switch** junction is a time t when the switch function $\Psi(t)$ (see (3)) changes sign and $\tau(t)$ has a discontinuity.)

Proof. At any time in I we have $\ddot{h} = K n \cdot d - G(x) = 0$ with $K > 0$ and $G(x) > 0$. Hence, $n \cdot d > 0$ and any **Switch** junction triggers a discontinuity of K and a non-zero discontinuity of $\ddot{h}$. This contradicts the fact that $\ddot{h}$ must stay zero in I . $\square$

Proposition 1 justifies our use, below, of the derivatives of K from Remark 1 that assume a constant $\tau(t)$.

The third derivative of h . We have assumed in Section 1.1 that $p_v \neq 0$, so that $d = \frac{p_v}{\|p_v\|}$: an expression that we can differentiate: $\ddot{h} = \ddot{n} \cdot v + 2K \dot{n} \cdot d + K n \cdot \dot{d} + \dot{K} n \cdot d$. Omitting the detailed derivation (which can be found in Hornus et al. (2026)), the components are:

$$\dot{d} = \frac{d}{dt} \left(\frac{p_v}{\|p_v\|} \right) = \frac{1}{\|p_v\|} f(d, p_r), \quad (6)$$

$$\dot{n} = \frac{\tan \gamma}{\|\bar{r}\|^3} f(\bar{r}, \bar{v}). \text{ Using (A.4), we obtain}$$

$$\ddot{n} = -3R \dot{n} + \frac{K \tan \gamma}{\|\bar{r}\|^3} f(\bar{r}, \bar{d}) - \frac{\tan \gamma}{\|\bar{r}\|^3} f(\bar{v}, \bar{r}),$$

$$\ddot{n} \cdot v = -3R \dot{n} \cdot v + \frac{K \tan \gamma}{\|\bar{r}\|^3} f(\bar{r}, \bar{d}) \cdot v - 0 \quad ((A.2) \text{ \& } (A.1))$$

$$= -3R \dot{n} \cdot v + \frac{K \tan \gamma}{\|\bar{r}\|^3} f(\bar{r}, \bar{v}) \cdot d \quad (\text{using (A.2)})$$

$$\ddot{n} \cdot v = -3R \dot{n} \cdot v + K \dot{n} \cdot d. \quad (7)$$

Substituting, we get:

$$\ddot{h} = 3(Kd - Rv) \cdot \dot{n} + (K\dot{d} + \dot{K}d) \cdot n. \quad (8)$$

Remark 2. The function $R : \mathbb{R}^+ \mapsto \mathbb{R}$ above is introduced to simplify expressions. It has the following derivatives:

$$R = \frac{\bar{r} \cdot \bar{v}}{\|\bar{r}\|^2}, \quad \dot{R} = \frac{\|\bar{v}\|^2 + K\bar{r} \cdot \bar{d}}{\|\bar{r}\|^2} - 2R^2.$$

Finding μ with the fourth derivative of h . We differentiate $\dot{h}$ and substitute (7) again:

$$\begin{aligned} h^{(4)} &= 3K\ddot{n} \cdot d \\ &+ \dot{n} \cdot \left(3(3R^2 - \dot{R})v + (4\dot{K} - 6RK)d + 4K\dot{d} \right) \\ &+ n \cdot \left(\ddot{K}d + 2\dot{K}\dot{d} + Kn \cdot \ddot{d} \right). \end{aligned} \quad (9)$$

In the equation above, one term is interesting: $\ddot{d}$. Expanding it, the function μ appears and we'll be able to compute μ from the other terms: From (6), we find

$$\ddot{d} = \frac{1}{\|p_v\|} \left((p_r \cdot \dot{d})d + 2(p_r \cdot d)\dot{d} - \mu f(d, n) \right). \quad (10)$$

We substitute (10) into (9) and, introducing a new symbol Ω , rewrite $h^{(4)}$ as $h^{(4)} = \Omega - \frac{K\mu}{\|p_v\|} f(d, n) \cdot n = 0$, where Ω does not depend on μ ; So $\mu = \frac{\Omega \|p_v\|}{K f(d, n) \cdot n}$ with

$$\begin{aligned} \Omega &= 3K\ddot{n} \cdot d \\ &+ \dot{n} \cdot \left(3(3R^2 - \dot{R})v + (4\dot{K} - 6RK)d + 4K\dot{d} \right) \\ &+ n \cdot \left(\ddot{K}d + 2\dot{K}\dot{d} + \frac{K}{\|p_v\|} \left((p_r \cdot \dot{d})d + 2(p_r \cdot d)\dot{d} \right) \right). \end{aligned}$$

On a glide-slope constrained arc, we use the expression above for μ in order to integrate the dynamics of the costate (2).

3. DIRECT AND INDIRECT METHODS

We give a very short overview of our implementation. A complete description is in Hornus et al. (2026).

To explore the different structures that can arise, we want to obtain a great many solutions from random initial conditions. Given a random initial position and velocity, we first obtain a direct solution (as a solution to a NLP), or generate a new random initial condition if the problem is not feasible. We then use the direct solution to determine one or more candidate structure(s) and parameters necessary for defining a shooting function. If we find a zero of a shooting function, we then check (numerically) that the resulting optimal trajectory satisfies all the constraints, in order to validate the candidate structure. If this succeeds, we record the resulting structure. Otherwise, we either retry with a higher-fidelity direct solution, or record a failure.

Evaluating the shooting function requires a candidate structure for the trajectory. A structure is a sequence of “junctions”: times (possibly with more parameters) at which the system dynamics changes. We have 3 kind of junctions: **Switch** junctions when the throttle changes. Their time is estimated from the throttle (control norm) in the direct solution. **Pointing** junctions (**start** and **stop**) bound a time interval wherein the pointing constraint is active. Their time is estimated from p_v (part of the dual of the dynamics constraints in the NLP). **Glide** junctions (**touch**, **start** and **stop**) describe the activation of the state constraint. We estimate that they occur at the local

maxima of p_h , the dual of the state constraint. The set of local *minima* of p_h below some threshold, do separate the local maxima into groups: we interpret singleton groups as **touch** junctions, and larger groups as **[start, stop]** intervals. By varying the threshold, we obtain several candidate structures for the state constraint activation. We try each candidate structure in turn, by assembling a specific shooting function for it. All these time estimates are fed directly into the shooting function(s).

4. NUMERICAL EXPERIMENTS

We use the martian lander described in Açıkmeşe and Ploen (2007). Since we are only interested in the structure of the optimal trajectory, we do not set a maximal fuel consumption. Still, our initial conditions are chosen reasonably so that the lander never uses, say, more than one metric ton of fuel.

The initial wet mass is $m_0 = 1905$ kg. The maximal thrust is $T = 16572.72$ N. Throttle range is given by $\tau_{\min} = 0.3$ and $\tau_{\max} = 0.8$. The parameter $\alpha \approx 5.0863 \times 10^{-4}$ s/m is computed from $\alpha^{-1} = g_e I_{sp} \cos \beta$ with $g_e = 9.807$ m/s², $I_{sp} = 225$ s and $\beta = 27$ deg. The pointing constraint parameter is $\theta^* = 60$ deg. The glide-slope angle is $\gamma = 10$ deg and martian gravity is $g_0 = 3.7114$ m/s².

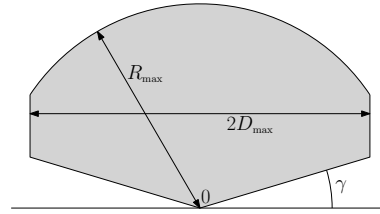


Fig. 2. The initial position r_0 is sampled in the gray volume.

The landing target is set at the origin. The initial position r_0 is generated within a maximal range and a maximal (3D) distance to the target. The sampling of r_0 is rotationally symmetric around the vertical axis above the target. We uniformly sample $w \in [0, D_{\max}]$: the horizontal distance (range) of r_0 to the origin ($w = \|\bar{r}_0\|$). Then the initial height $r_{0,z}$ is uniformly sampled in $[w \tan \gamma, \sqrt{R_{\max}^2 - w^2}]$. A choice of $R_{\max} \geq \frac{D_{\max}}{\cos \gamma}$ guarantees that this interval is nonempty. In our implementation we choose $D_{\max} = 2000$ m and $R_{\max} = 1.5 \times \frac{D_{\max}}{\cos \gamma}$. For the initial velocity, we uniformly choose $v_{0,z}$ in $[-100, 25]$ m/s and $\bar{v}_0$ in $[-100, 100]^2$ m/s.

Before computing a direct solution (Section 3), we perform a simple sanity check that detects (and rejects) some unfeasible initial state: We consider a 1D motion along the line with direction $n(r_0)$ and test that the maximal thrust along the normal to the glide-slope cone is enough to avoid crashing into the cone. We write $v_n = v_0 \cdot \hat{n}(r_0)$, $r_n = r_0 \cdot \hat{n}(r_0)$ and $a = \frac{T_{\max}}{m_0} - g_0 \cos \gamma$. The test fails if $v_n^2 > 2ar_n$. A new initial state is sampled as long as the current one fails the sanity check.

4.1 Statistics

We tested our indirect solver on 1100000 random initial conditions for which IpOpt finds a direct solution. For 32 of these, our software failed to find a satisfying structure (while IpOpt found a direct solution at the finest allowed time-grid discretization).

Count	Structure	Freq. (%)
483581	M θ m θ m M	439.632
246409	M θ M m M	224.015
86573	m θ m M	78.705
58440	M m M	53.129
55009	M $\star$ M m M $\star$ M	50.010
45907	M $\star$ M m M	41.735
31537	M m M $\star$ M	28.671
25648	m M	23.317
19841	M M θ $\star$ M θ M m M	18.038
9772	M	8.884
8855	M θ M	8.050
6472	M θ M m M $\star$ M	5.884
4663	M $\star$ M m M M γ	4.239

Table 1. The 13 most frequent “successfully” recovered structures. ‘M’ stands for a bang-max arc ($\tau = \tau_{\max}$), ‘m’ means bang-min. We adjoin ‘ γ ’ when the arc is state-constrained and ‘ θ ’ when it is pointing-constrained. The star $\star$ stands for a **Glide touch** junction.

Among the failures we are able to find an indirect solution for one, at the cost of a slightly longer computation time, used to compute a higher resolution direct solution. We believe that the 31 initial conditions with failure probably have, in fact, no solution: For these, the lander is initially moving quickly towards the glide-slope cone and the trajectories show a **Glide touch** junction before turning back towards the landing target. The indirect solution shows a similar trajectory but dips below the glide-slope cone by at most a few millimeters, *i.e.* the state constraint can not be satisfied at the contact with the cone. The direct and indirect trajectory are similar in shape but are far from each other. We conjecture that these initial conditions have no feasible solutions but are very close to conditions with a feasible solution, and IpOpt has enough numerical freedom to find a false solution. We then verified that at higher time-grid resolution, IpOpt fails indeed to find a solution.

The 1099968 successfully computed indirect solutions exhibit 56 different trajectory structures. Table 1 shows the 13 most frequent structures, which account for 98.431% of the trajectories. Among all trajectories, 16.3% have at least one **Glide** junction. Remarkably, glide arcs of *positive* duration appear in 0.750% of all trajectories. Figure 3 shows a trajectory that exhibits two **Switch** junctions, two **Pointing** junctions and two **Glide touch** junctions. The statistics in Table 2 focus on each constraint independently. We want to highlight the wide variety of structures generated by the state constraint. The two trajectories with two pointing-constrained arcs are probably wrong and due to some bug in our code; an investigation is expected.

5. ANALYSIS OF GLIDE-SLOPE ARCS

In this section, we look at the possible interactions between the glide-slope (state) constraint and the pointing (control) constraint. We argue that not much can happen while the lander is sliding along the glide-slope cone.

Proposition 2. Assume that $\gamma \in [0, \pi/2)$ and $\theta^* \in (\gamma, \pi - \gamma)$. Assume that $\|p_v\| > 0$. Let $I = (a, b)$, $a < b$, be a time interval along which the glide-slope constraint is active. If

Count	Structure	Freq. (%)
920997		837.294
89810	$\star$	81.648
56886	$\star \star$	51.716
23606	$\star \theta$	21.461
4749	$\star \bar{\gamma}$	4.317
3251	$\bar{\gamma}$	2.956
410	$\star \star \theta$	0.373
108	$\gamma \star$	0.098
44	γ	0.040
40	$\gamma \bar{\gamma}$	0.036
32	$\gamma \star \theta$	0.029
10	$\star \gamma \star$	0.009
8	$\star \star \star \theta$	0.007
6	$\star \gamma \star \theta$	0.005
3	$\star \gamma$	0.003
3	$\star \star \bar{\gamma}$	0.003
3	$\star \star \star$	0.003
1	$\star \star \gamma$	0.001
1	$\star \theta \star$	0.001

A. Statistics on the glide-slope state constraint activation. The symbol $\bar{\gamma}$ represents a state-constrained arc that lasts until the final time t_f .

Count	Structure	Freq. (%)
837042	$\theta \dots$	760.969
238838	$\dots$	217.132
24045	$\dots \theta \star \dots$	21.860
23	$\dots \theta$	0.021
18	$\theta \star \dots$	0.016
2	$\theta \theta$	0.002

B. Statistics on the pointing control constraint activation. The ellipsis represents some arc(s) of positive length that are not pointing-constrained.

Count	Structure	Freq. (%)
961400	M m M	874.025
113666	m M	103.336
24902	M	22.639

C. Statistics on the bang structure.

Table 2. Statistics for each constraint.

there is a **Pointing start** junction at time $t^* \in I$, then we must have $\dot{\theta}(t^*) = \ddot{\theta}(t^*) = 0$.

Proof. This is a proof sketch that omits most derivations. The details are in Hornus et al. (2026). We consider the third derivative of h and prove that if the pointing constraint becomes active at time $t^* \in I$, then, unless $\dot{\theta}(t^*) = \ddot{\theta}(t^*) = 0$, $\ddot{h}$ undergoes a non-zero discontinuity that contradicts the existence of the interval I , along which $\ddot{h}$ must stay zero. It is convenient to work in cylindrical and spherical coordinates. The position is parameterized as

$$r = \begin{pmatrix} \rho e^{i\varphi} \\ z \end{pmatrix} = \begin{pmatrix} \rho \cos \varphi \\ \rho \sin \varphi \\ z \end{pmatrix}. \quad (11)$$

We also define spherical coordinate angles for p_v in the classical orthonormal basis $(e_\sigma, e_\theta, e_\psi)$:

$$e_\sigma = \begin{pmatrix} \sin \theta e^{i\psi} \\ \cos \theta \end{pmatrix}, \quad e_\theta = \begin{pmatrix} \cos \theta e^{i\psi} \\ -\sin \theta \end{pmatrix}, \quad e_\psi = \begin{pmatrix} i e^{i\psi} \\ 0 \end{pmatrix}. \quad (12)$$

Using $p_r = -\dot{p}_v$, we find

$$p_v = \sigma e_\sigma \quad \text{and} \quad p_r = -\dot{\sigma} e_\sigma - \sigma \dot{\theta} e_\theta - \sigma \dot{\psi} \sin \theta e_\psi. \quad (13)$$

The direction of the control is given by $d = \begin{pmatrix} \sin \check{\theta} e^{i\check{\psi}} \\ \cos \check{\theta} \end{pmatrix}$ with

$$\check{\theta} = \begin{cases} \theta & \text{if } 0 \leq \theta \leq \theta^* \text{ (inactive pointing constraint)} \\ \theta^* & \text{if } \theta^* \leq \theta < \pi \text{ (active pointing constraint)} \end{cases} \quad (14)$$

The dynamics give us $\dot{v} = K d + \begin{pmatrix} 0 \\ 0 \\ -g_0 \end{pmatrix}$.

Now let us compute derivatives of h :

$$\begin{aligned} h &= z - \rho \tan \gamma = 0, \\ \dot{h} &= \dot{z} - \dot{\rho} \tan \gamma = 0. \end{aligned} \quad (15)$$

Substituting the dynamics in $\dot{z} - \dot{\rho} \tan \gamma$ yields

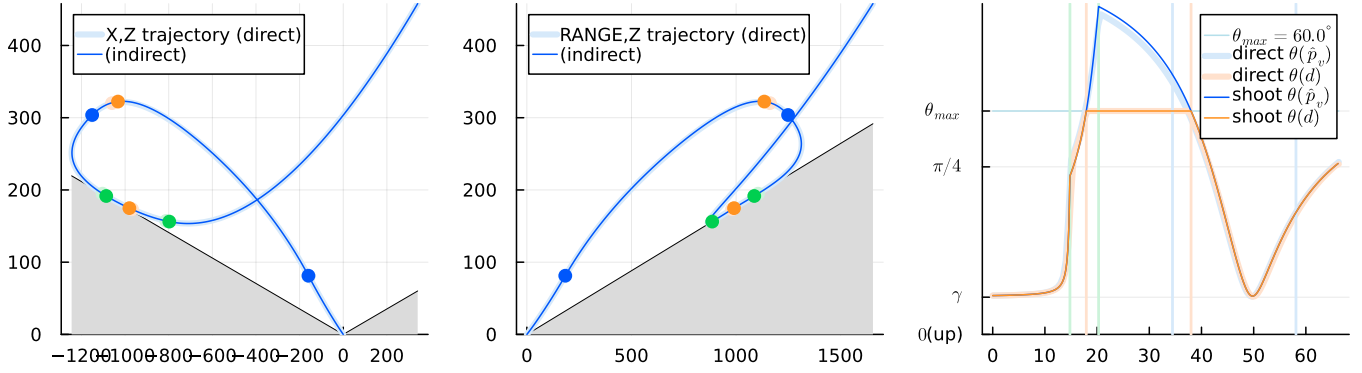


Fig. 3. A trajectory with structure $[M \star M \text{ M}\theta \star \text{M}\theta \text{ m}\theta \text{ m } M]$, $r_0 \approx [341.22241899750037, -1618.4981582743467, 457.94072899950186]$, $v_0 \approx [-86.35707101999044, 80.6611492151617, -44.41706527884088]$. Blue dots: **Switch** junctions, orange: **Pointing** and green: **Glide** junctions.

$$\ddot{h} = K \cos \check{\theta} - g_0 - (\rho \dot{\varphi}^2 + K \sin \check{\theta} \cos(\psi - \varphi)) \tan \gamma = 0. \quad (16)$$

Differentiating once more, one gets (the first line uses $\dot{K} = \alpha K^2$ and the second line comes from differentiating $\rho \dot{\varphi}^2$, and substituting the dynamics again):

$$\begin{aligned} \ddot{h} &= \alpha K (g_0 + \rho \dot{\varphi}^2 \tan \gamma) \\ &+ \left(3 \rho \dot{\varphi}^2 + K \sin \check{\theta} (\dot{\psi} - 3 \dot{\varphi}) \sin(\psi - \varphi) \right) \tan \gamma \\ &- K \left(\sin \check{\theta} + \cos \check{\theta} \cos(\psi - \varphi) \tan \gamma \right) \frac{d}{dt} \check{\theta} = 0 \end{aligned} \quad (17)$$

where (see (14))

$$\frac{d}{dt} \check{\theta} = \begin{cases} \dot{\theta} & \text{if } 0 \leq \theta < \theta^* \text{ (inactive pointing constraint)} \\ 0 & \text{if } \theta^* < \theta < \pi \text{ (active pointing constraint)} \end{cases} \quad (18)$$

Define $A(\theta, \phi, \psi) \equiv \sin \theta + \cos \theta \tan \gamma \cos(\psi - \varphi)$. We see that everything in $\ddot{h}$ is continuous with respect to time, except the last term, that has, at $\theta = \theta^*$, a jump with size $KA(\theta^*, \phi, \psi)\dot{\theta}$; hence $\ddot{h}(t)$ can be identically zero in a neighborhood of a time such that θ is smaller than θ^* before t^* and larger than θ^* after t^* only if

$$KA(\theta^*, \phi, \psi)\dot{\theta} = 0. \quad (19)$$

On the one hand, $K > 0$. On the other hand, our assumption that $[\gamma \in [0, \pi/2)$ and $\theta^* \in (\gamma, \pi - \gamma)]$ implies $A(\theta^*, \phi, \psi) > 0$. So, if the pointing constraint is activated, we must have $\dot{\theta}(t^*) = 0$. Assume now that $\ddot{\theta}(t^*) \neq 0$. Since $\theta(t) < \theta^*$ for $t < t^*$, $\ddot{\theta}(t^*)$ must be negative and the pointing constraint can not be activated at time t^* . So, if the pointing constraint is activated, we must have $\ddot{\theta}(t^*) = 0$. $\square$

Remark 3. Proposition 2 suggests that it is very unlikely that the pointing constraint gets activated inside a state-constrained time interval. We have never seen, experimentally, both constraints active at the same time along an interval of positive length; we conjecture that this is not possible. Another way to have both constraints active would be to activate the glide-slope constraint while the pointing control constraint is already active. In this case it is also possible to derive the value of μ along such an arc (Hornus et al. (2026)), but we have never seen it happen in our tests.

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Appendix A. THE AUXILIARY FUNCTION f

For any vectors a and b , define $f(a, b) \equiv (a \cdot b)a - \|a\|^2 b$. Using $f(a, b) = -\|a\|^2 (b - (\hat{a} \cdot b)\hat{a})$, we see that f projects b onto the plane orthogonal to a , then scale it by $-\|a\|^2$. The function f simplifies some calculus, as follows:

$$f(a, a) = 0, \quad (A.1)$$

$$f(a, b) \cdot c = f(a, c) \cdot b \Rightarrow f(a, b) \cdot a = 0, \quad (A.2)$$

$$f(a, b) \cdot b = f(b, a) \cdot a \leq 0. \quad (A.3)$$

We define $j(a, b) \equiv \frac{d}{dt} f(a, b)$. Then,

$$\begin{aligned} j(a, b) &= f(a, \dot{b}) + (\dot{a} \cdot b)a + (a \cdot b)\dot{a} - 2(a \cdot \dot{a})b, \\ j(a, \dot{a}) &= f(a, \ddot{a}) - f(\dot{a}, \dot{a}). \end{aligned} \quad (A.4)$$